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Vertex algebras generated by Lie algebras

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Abstract

In this paper we introduce a notion of vertex Lie algebra U, in a way a "half" of vertex algebra structure sufficient to construct the corresponding local Lie algebra $\mathscr{L}(U)$ and a vertex algebra $\mathscr{V}(U)$. We show that we may consider U as a subset $U \subset \mathscr{V}(U)$ which generates $\mathscr{V}(U)$ and that the vertex Lie algebra structure on U is induced by the vertex algebra structure on $\mathscr{V}(U)$. Moreover, for any vertex algebra V a given homomorphism $U \to V$ of vertex Lie algebras extends uniquely to a homomorphism $\mathscr{V}(U) \to V$ of vertex algebras. In the second part of paper we study under what conditions on structure constants one can construct a vertex Lie algebra U by starting with a given commutator formula for fields. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

One way of seeing a vertex algebra V is as a vector space with infinitely many bilinear multiplications $u_n v$, $n \in \mathbb{Z}$, which correspond to "normal order products" $u(z)_n v(z)$ of fields $u(z)$ and $v(z)$ associated to vectors u and v. For two fields there is a formula in which the commutator is expressed in terms of products $u(z)_n v(z)$ only for $n \geq 0$. So any vector space $U \subset V$ closed for "positive" multiplications will look like some kind of Lie algebra.

In this paper we define a vertex Lie algebra as a vector space U given infinitely many bilinear multiplications $u_n v$, $n \in \mathbb{N}$, satisfying a Jacobi identity and a skew symmetry in terms of given derivation D. In particular, any vertex algebra is a vertex Lie algebra if we forget the multiplications other than for $n \geq 0$. This structure is sufficient to construct a Lie algebra $\mathscr{L}(U)$ generated by vectors u_n, $u \in U$, $n \in \mathbb{Z}$, such that the

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commutator formula for fields with $u(z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$ defines the commutator in $\mathscr{L}(U)$. This Lie algebra has the obvious decomposition $\mathscr{L}(U) = \mathscr{L}_-(U) \oplus \mathscr{L}_+(U)$ and the induction by a trivial $\mathscr{L}_+(U)$-module gives a generalized Verma $\mathscr{L}(U)$-module

$$\mathscr{V}(U) = \mathscr{U}(\mathscr{L}(U)) \otimes_{\mathscr{U}(\mathscr{L}_+(U))} \mathbb{C} \cong \mathscr{U}(\mathscr{L}_-(U)),$$

where $\mathscr{U}$ stands for the universal enveloping algebra of a given Lie algebra.

We show that $\mathscr{V}(U)$ is a vertex algebra and that we may consider U as a subset $U \subset \mathscr{V}(U)$ which generates $\mathscr{V}(U)$. So an "abstract" U turns to be as in the "concrete" motivating example at the beginning. Moreover, for any vertex algebra V a given homomorphism $U \to V$ of vertex Lie algebras extends uniquely to a homomorphism $\mathscr{V}(U) \to V$ of vertex algebras. Because of this universal property we call $\mathscr{V}(U)$ the universal enveloping vertex algebra of U. These constructions are modeled after and apply to the well known examples of vertex (super)algebras associated to affine Lie algebras, Virasoro algebra and Neveu–Schwarz algebra.

In the second part of this paper we study under what conditions on structure constants one can construct a vertex algebra by starting with a given commutator formula for fields, or equivalently, by starting with a given singular part of operator product expansion for fields. In the case of a commutator formula closed for a set S of quasi-primary fields we give explicit necessary and sufficient conditions for the existence of universal vertex algebra $\mathscr{V}(\langle S\rangle)$ generated by S.

This work rests on the inspiring results of Hai-sheng Li in [12] and in a way a complementary theorem on generating fields in [5, 14], but it goes without saying that many ideas used here stem from [1, 2, 6–8], to mention just a few. Li's point of view on generating fields gives a natural framework for studying modules and it was tempting to see how some of his Lie-theoretic arguments could be extended to a more general setting. The key technical point is the observation that a direct proof that the local algebra $\mathscr{L}(V)$ of vertex algebra V is a Lie algebra involves just a "positive half" of both the Jacobi identity and the skew symmetry, i.e., to be more precise, involves only the principal part of the formal Laurent series which appear in these relations. In a similar way only a "positive half" of the commutator formula implies the "positive half" of the Jacobi identity. Some of these arguments are essentially a simpler copy of Li's arguments for vertex (super)algebras and work just the same in the vertex superalgebra case.

2. Generating fields for vertex algebras

For a $\mathbb{Z}_2$-graded vector space $W = W^0 + W^1$ we write $|u| \in \mathbb{Z}_2$, a degree of u, only for homogeneous elements: $|u| = 0$ for an even element $u \in W^0$ and $|u| = 1$ for an odd element $u \in W^1$. For any two $\mathbb{Z}_2$-homogeneous elements u and v we define $\varepsilon_{u,v} = (-1)^{|u||v|} \in \mathbb{Z}$.

By following [12], a vertex superalgebra V is a $\mathbb{Z}_2$-graded vector space $V = V^0 + V^1$ equipped with a specified vector $\mathbf{1}$ called the vacuum vector, a linear operator D on

V called the derivation and a linear map

$$V \to (\mathrm{End}\, V)[[z^{-1}, z]], \qquad v \mapsto Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$$

satisfying the following conditions for $u, v \in V$:

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large;} \tag{2.1}$$

$$[D, Y(u,z)] = Y(Du,z) = \frac{\mathrm{d}}{\mathrm{d}z} Y(u,z); \tag{2.2}$$

$$Y(\mathbf{1},z) = \mathrm{id}_V \quad \text{(the identity operator on } V\text{);} \tag{2.3}$$

$$Y(u,z)\mathbf{1} \in (\mathrm{End}\, V)[[z]] \quad \text{and} \quad \lim_{z\to 0} Y(u,z)\mathbf{1} = u. \tag{2.4}$$

For $\mathbb{Z}_2$-homogeneous elements $u, v \in V$ the Jacobi identity holds:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u,z_1)Y(v,z_2) - \varepsilon_{u,v} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v,z_2)Y(u,z_1)$$

$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v, z_2). \tag{2.5}$$

Finally, for any $\mathbb{Z}_2$-homogeneous $u, v \in V$ and $n \in \mathbb{Z}$ we assume

$$|u_n v| = |u| + |v| \tag{2.6}$$

(i.e., $u_n v$ is homogeneous and $|u_n v| = |u| + |v|$).

Sometimes we will emphasize that $Y(u,z)$ is pertinent to a vertex superalgebra V by writing $Y_V(u,z)$.

In the definition of vertex superalgebra, VSA for short, condition (2.4) can be equivalently replaced by the condition $Y(u,z)\mathbf{1} = \mathrm{e}^{zD}u$ for all $u \in V$, and both are called the creation property. In a way the creation property is a special case of the skew symmetry

$$Y(u,z)v = \varepsilon_{u,v} \mathrm{e}^{zD} Y(v,-z)u$$

which holds for all homogeneous $u, v \in V$.

Note that in the case when $V = V^0$, i.e., when all vectors are even, all $\mathbb{Z}_2$-grading conditions become trivial and we speak of a vertex algebra, VA for short. In general, for a $\mathbb{Z}_2$-graded vector space W the vector spaces

$$\mathrm{End}\, W = (\mathrm{End}\, W)^0 \oplus (\mathrm{End}\, W)^1,$$

$$(\mathrm{End}\, W)[[z^{-1}, z]] = (\mathrm{End}\, W)^0[[z^{-1}, z]] \oplus (\mathrm{End}\, W)^1[[z^{-1}, z]]$$

are $\mathbb{Z}_2$-graded as well and our assumption (2.6) implies that for a homogeneous element $u \in V$ the operators u_n are homogeneous for all $n \in \mathbb{Z}$, that $Y(u,z)$ is homogeneous and

$$|u| = |u_n| = |Y(u,z)|.$$

This together with (2.4), (2.3) and (2.2) implies that $\mathbf{1} \in V$ is even and that $D \in \mathrm{End}\, V$ is even. On the other hand, $\mathbf{1} \in V$ is even together with (2.1), (2.4) and (2.5) implies

(2.2) for D defined by $Du = u_{-2}\mathbf{1}$; relation (2.3) follows as well. For this reason we define a homomorphism $\varphi : V \to U$ of two vertex superalgebras as a $\mathbb{Z}_2$-grading preserving linear map φ such that

$$\varphi(u_n v) = (\varphi(u))_n(\varphi(v))$$

for all $u, v \in V$, $n \in \mathbb{Z}$; as a consequence we have relations

$$\varphi(\mathbf{1}) = \mathbf{1}, \qquad \varphi D = D\varphi.$$

For a subset $U \subset V$ we denote by $\langle U \rangle$ a vertex superalgebra generated by U, i.e., the smallest vertex superalgebra containing the set U.

A module M for a vertex superalgebra V is a $\mathbb{Z}_2$-graded vector space $M = M^0 + M^1$ equipped with an even linear operator $D \in (\operatorname{End} M)^0$ and a linear map

$$V \to (\operatorname{End} M)[[z^{-1}, z]], \qquad v \mapsto Y_M(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$$

satisfying the following conditions for $u, v \in V$ and $w \in M$:

$$u_n w = 0 \quad \text{for} \quad n \in \mathbb{Z} \text{ sufficiently large;} \tag{2.7}$$

$$[D, Y_M(u,z)] = Y_M(Du,z) = \frac{\mathrm{d}}{\mathrm{d}z} Y_M(u,z); \tag{2.8}$$

$$Y_M(\mathbf{1},z) = \mathrm{id}_M \quad \text{(the identity operator on } M\text{)}. \tag{2.9}$$

For $\mathbb{Z}_2$-homogeneous elements $u, v \in V$ the Jacobi identity holds:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u,z_1)Y_M(v,z_2) - \varepsilon_{u,v} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(v,z_2)Y_M(u,z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u,z_0)v,z_2). \tag{2.10}$$

Finally, for any $\mathbb{Z}_2$-homogeneous $u \in V$, $w \in M$ and $n \in \mathbb{Z}$ we assume

$$|u_n w| = |u| + |w| \tag{2.11}$$

(i.e., $u_n w$ is homogeneous and $|u_n w| = |u| + |w|$).

Clearly, V is a V-module with $Y_V(u,z) = Y(u,z)$, and, as before, for a V-module M we have

$$|u| = |u_n| = |Y_M(u,z)|.$$

Let M be a V-module and $u, v \in V$ homogeneous elements. Set $u(z) = Y_M(u,z)$ and $v(z) = Y_M(v,z)$. As a consequence of the definition of V-module M we have the normal order product formula and the locality: The normal order product formula states that the field $Y_M(u_n v, z)$ equals

$$u(z)_n v(z) = \operatorname{Res}_{z_1}((z_1 - z)^n u(z_1)v(z) - (-1)^{|u(z)||v(z)|}(-z + z_1)^n v(z)u(z_1)). \tag{2.12}$$

The locality property states that for some $N=N(u,v)\in\mathbb{N}$

$$(z_1-z_2)^N u(z_1)v(z_2)=(-1)^{|u(z)||v(z)|}(z_1-z_2)^N v(z_2)u(z_1). \tag{2.13}$$

In the case when formal Laurent series $u(z)$ and $v(z)$ satisfy relation (2.13) for some $N=N(u(z),v(z))\in\mathbb{N}$, we say that $u(z)$ and $v(z)$ are local to each other.

Hai-sheng Li proved [12, Proposition 2.2.4] that in the definition of vertex superalgebra the Jacobi identity (2.5) can be equivalently substituted by the locality property (2.13).

Let M be a $\mathbb{Z}_2$-graded vector space $M=M^0+M^1$ equipped with an even linear operator $D\in(\operatorname{End}M)^0$. Following [12] define a vertex operator on M as a homogeneous formal Laurent series $v(z)=\sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ in $(\operatorname{End}M)[[z^{-1},z]]$ such that the property (2.7) holds for all $w\in W$, that $v(z)$ is local with itself (i.e., that (2.13) holds for $u(z)=v(z)$) and that

$$[D,v(z)]=\frac{\mathrm{d}}{\mathrm{d}z}v(z). \tag{2.14}$$

We shall also say that a linear combination of vertex operators on M is a vertex operator on M.

It is clear that giving a map $Y(\cdot,z)$ is equivalent to giving infinitely many bilinear multiplications $u_n v$, $n\in\mathbb{Z}$. Let $F(M)$ be a space of formal Laurent series $v(z)$ in $(\operatorname{End}M)[[z^{-1},z]]$ such that the property (2.7) holds for all $w\in W$. Then $F(M)$ is $\mathbb{Z}_2$-graded and on $F(M)$ bilinear multiplications $u(z)_n v(z)$, $n\in\mathbb{Z}$, given by (2.12) are well defined. Moreover, Hai-sheng Li proved the following theorem [12, Corollary 3.2.11, Theorem 3.2.10]:

Theorem 2.1. *Let M be any $\mathbb{Z}_2$-graded vector space equipped with an even linear operator D and let U be any set of mutually local homogeneous vertex operators on M. Let $\langle U\rangle$ be the subspace of $F(M)$ generated by U and $I(z)=\mathrm{id}_M$ under the vertex operator multiplication* (2.12). *Then $\langle U\rangle$ is a vertex superalgebra with the vacuum vector $\mathbf{1}=I(z)$ and the derivation $D=\mathrm{d}/\mathrm{d}z$. Moreover, M is a $\langle U\rangle$-module.*

Clearly this theorem implies that for a vertex superalgebra V the set of fields $\{Y(u,z)\mid u\in V\}$ is a vertex superalgebra, and by construction it is clear that $u\mapsto Y(u,z)$ is an isomorphism. Because of this isomorphism we shall sometimes say that for a subset $U\subset V$ the vertex superalgebra $\langle U\rangle\subset V$ is generated by the set of fields $\{Y(u,z)\mid u\in U\}$.

For the following theorem see [5, 8, 13, 17]. The theorem was proved in [14] for graded vertex algebras, the results in [12] allow us to extend the proof to the vertex superalgebra case:

Theorem 2.2. *Let V be a $\mathbb{Z}_2$-graded vector space $V=V^0+V^1$ equipped with an even linear operator $D\in(\operatorname{End}V)^0$ and an even vector $\mathbf{1}$ such that $D\mathbf{1}=0$. Let U be a*

$\mathbb{Z}_2$-graded subspace of V given a linear map

$$Y: U \to (\mathrm{End}\, V)[[z, z^{-1}]], \qquad u \mapsto Y(u,z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$$

such that $\{Y(u,z) \mid u \in U^0 \cup U^1\}$ is a set of mutually local vertex operators on V satisfying the following two conditions:

$$Y(u,z)\mathbf{1} \in (\mathrm{End}\, V)[[z]] \quad \text{and} \quad \lim_{z\to 0} Y(u,z)\mathbf{1} = u; \tag{2.15}$$

$$V = \mathrm{span}\{u^{(1)}_{n_1} \cdots u^{(k)}_{n_k}\mathbf{1} \mid k \in \mathbb{N},\ n_i \in \mathbb{Z},\ u^{(i)} \in U^0 \cup U^1\}. \tag{2.16}$$

Then Y extends uniquely into a vertex superalgebra with the vacuum vector **1** *and the derivation D.*

Proof. Let W be the $\mathbb{Z}_2$-graded space of all vertex operators $a(z)$ on V such that $a(z)$ and $Y(u,z)$ are mutually local for each $u \in U^0 \cup U^1$ and that $a(z)\mathbf{1}$ is a power series in z. Define a linear map

$$\phi: W \to V, \qquad a(z) \mapsto a_{-1}\mathbf{1}.$$

Note that our assumptions imply $Y(u,z) \in W$ for all $u \in U$. Hence by (2.15)

$$U \subset \phi(W) \subset V.$$

Also note that ϕ preserves the $\mathbb{Z}_2$-grading and that $\phi(W) \subset V$ is a $\mathbb{Z}_2$-graded subspace.

Step 1: *ϕ is injective.* Assume that $a(z) \in W$ is homogeneous and $a_{-1}\mathbf{1} = 0$. Since $a(z)\mathbf{1}$ is a power series in z, (2.14) and $D\mathbf{1} = 0$ imply

$$a(z)\mathbf{1} = e^{zD} a_{-1}\mathbf{1} = 0. \tag{2.17}$$

Let $X = \{v \in V \mid a(z)v = 0\}$ and let $v \in X$, $u \in U^0 \cup U^1$. Then there exists $N \in \mathbb{N}$ such that

$$(z_1 - z_2)^N a(z_1) Y(u,z_2) v = (-1)^{|a(z)||u|} (z_1 - z_2)^N Y(u,z_2) a(z_1) v = 0.$$

Hence $a(z_1)u_n v = 0$ and $u_n X \subset X$. Since (2.17) implies $\mathbf{1} \in X$, it follows from (2.16) that $X = V$. Hence $a(z) = 0$ and ϕ is injective.

Step 2: Set $X = \phi(W)$. Since ϕ is injective we can define

$$Y: X \to (\mathrm{End}\, V)[[z, z^{-1}]] \quad \text{by} \quad Y(v,z) = \phi^{-1}(v)$$

for $v \in X$. By (2.15) the two meanings of $Y(u,z)$ for $u \in U$ denote the same series. *We claim that $X = \phi(W) = V$, i.e., we have well defined vertex operators $Y(v,z)$ for all $v \in V$.*

For $n \in \mathbb{Z}$ and homogeneous elements $u \in U$, $v \in X$, define $Y(u_n v, z)$ by using the product (2.12):

$$Y(u_n v, z) = (Y(u,z))_n (Y(v,z)).$$

Since $Y(u,z)$ and $Y(v,z)$ are in W, it follows from [12, Lemmas 3.1.4, 3.1.8, Proposition 3.2.7] that $Y(u_nv,z)$ is a vertex operator on V. Again by [12, Propostion 3.2.7] vertex operators $Y(u_nv,z)$ and $Y(w,z)$ are mutually local for each $w\in U^0\cup U^1$. It follows from (2.12) that $Y(u_nv,z)\mathbf{1}$ is a power series in z. Hence $Y(u_nv,z)\in W$. It follows from (2.12) (cf. [12, (3.1.9)]) that

$$\phi(Y(u_nv,z))=(u_nv)_{-1}\mathbf{1}=u_nv_{-1}\mathbf{1}=u_nv.$$

Hence $u\in U,\ v\in X,\ n\in\mathbb{Z}$ implies

$$u_nv\in X. \tag{2.18}$$

Since $\mathrm{id}_V z^0\in W$ and $\mathrm{coeff}_{z^0}(\mathrm{id}_V z^0)\mathbf{1}=\mathbf{1}$, we have

$$\mathbf{1}\in X \tag{2.19}$$

and $Y(\mathbf{1},z)=\mathrm{id}_V z^0=\mathrm{id}_V$. Now (2.16), (2.18), (2.19) imply $X=V$, i.e., we have well defined vertex operators $Y(v,z)$ for all $v\in V$.

Step 3: *$Y(u,z)$ and $Y(v,z)$ are mutually local for all pairs $u,v\in V$.* Fix a homogeneous element $v^{(0)}\in V$ and set

$$X=\mathrm{span}\{v\in V\mid Y(v,z) \text{ and } Y(v^{(0)},z) \text{ and mutually local}\}.$$

Clearly $\{\mathbf{1}\}\cup U\subset X$. For $n\in\mathbb{Z}$ and homogeneous elements $u\in U,\ v\in X$ by definition $Y(u_nv,z)=(Y(u,z))_n(Y(v,z))$, so [12, Proposition 3.2.7] implies that $u_nv\in X$. Hence (2.16) again implies $X=V$ as required.

We have showed that V satisfies the conditions of [12, Proposition 2.2.4], so V is a vertex superalgebra. □

3. Vertex Lie algebras

Let V be a vertex superalgebra. By taking $\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}z_0^n$ of the Jacobi identity for V or the Jacobi identity for a V-module M, that is the coefficients of $z_0^{-n-1}z_1^{-1}$, we get the normal order product formula (2.12). These relations are the components of associator formula obtained by taking the residue Res_{z_1} of the Jacobi identity:

$$\begin{aligned}
&Y(Y(u,z_0)v,z_2)-Y(u,z_0+z_2)Y(v,z_2)\\
&\quad=-\varepsilon_{u,v}\,\mathrm{Res}_{z_1}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)\\
&\quad=\varepsilon_{u,v}Y(v,z_2)\{Y(u,z_2+z_0)-Y(u,z_0+z_2)\}.
\end{aligned}$$

By taking the residue Res_{z_0} of the Jacobi identity we get the commutator formula

$$[Y(u,z_1),Y(v,z_2)] = \mathrm{Res}_{z_0} z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y(Y(u,z_0)v,z_2)$$

$$= \sum_{i\geq 0} \frac{(-1)^i}{i!}\left(\frac{\mathrm{d}}{\mathrm{d}z_1}\right)^i z_2^{-1}\delta(z_1/z_2)Y(u_iv,z_2). \tag{3.1}$$

We can write these identities for components u_m and v_n of $Y(u,z)$ and $Y(v,z)$:

$$u_mv_n - \varepsilon_{u,v}v_nu_m = \sum_{i\geq 0}\binom{m}{i}(u_iv)_{n+m-i}.$$

In general, if we take $\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_0^kz_1^mz_2^n$ of the Jacobi identity applied to a vector w, that is the coefficients of $z_0^{-k-1}z_1^{-m-1}z_2^{-n-1}$, we get for components of vertex operators the identities

$$\sum_{i\geq 0}(-1)^i\binom{k}{i}(u_{m+k-i}(v_{n+i}w) - \varepsilon_{u,v}(-1)^k v_{n+k-i}(u_{m+i}w))$$

$$= \sum_{i\geq 0}\binom{m}{i}(u_{k+i}v)_{m+n-i}w. \tag{3.2}$$

These relations hold for all $k,m,n\in\mathbb{Z}$, but it should be noticed that for $k,m,n\in\mathbb{N}$ these relations involve only indices in $\mathbb{N}$. In a way the purpose of this paper is to study the consequences of this "half" of the Jacobi identity.

Let A and B be two formal Laurent series (in possibly several variables $z_0,z_1,\ldots$). We shall write $A\simeq B$ if the principal parts of A and B are equal. For example, $A(z_0,z_1,z_2)\simeq B(z_0,z_1,z_2)$ means that the coefficients of $z_0^{-k-1}z_1^{-m-1}z_2^{-n-1}$ in A equal the coefficients in B for all $k,m,n\in\mathbb{N}$. In particular, the set of relations (3.2) for all $k,m,n\in\mathbb{N}$ is equivalent to the "half Jacobi identity" (3.6) written below. In a similar way we shall speak of the half commutator formula or the half associator formula. From the way they were obtained, it is clear they are a subset of the half Jacobi identity viewed by components.

With the above notation we define a vertex Lie superalgebra U as a $\mathbb{Z}_2$-graded vector space $U=U^0+U^1$ equipped with an even linear operator D on U called the derivation and a linear map

$$U\to z^{-1}(\mathrm{End}\,U)[[z^{-1}]],\qquad v\mapsto Y(v,z)=\sum_{n\geq 0}v_nz^{-n-1}$$

satisfying the following conditions for $u,v\in U$:

$$u_nv=0 \quad \text{for } n \text{ sufficiently large;} \tag{3.3}$$

$$[D,Y(u,z)] = Y(Du,z) = \frac{\mathrm{d}}{\mathrm{d}z}Y(u,z). \tag{3.4}$$

For $\mathbb{Z}_2$-homogeneous elements $u, v \in V$ the half skew symmetry holds:

$$Y(u,z)v \simeq \varepsilon_{u,v} e^{zD} Y(v,-z)u. \tag{3.5}$$

For $\mathbb{Z}_2$-homogeneous elements $u, v \in V$ the half Jacobi identity holds:

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2) - \varepsilon_{u,v} z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)$$
$$\simeq z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2). \tag{3.6}$$

Finally, for any $\mathbb{Z}_2$-homogeneous $u, v \in U$ and $n \geq 0$ we assume

$$|u_n v| = |u| + |v| \tag{3.7}$$

(i.e., $u_n v$ is homogeneous and $|u_n v| = |u| + |v|$).

Sometimes we will emphasize that $Y(u,z)$ is pertinent to a vertex Lie superalgebra U by writing $Y_U(u,z)$.

Loosely speaking, a vertex Lie superalgebra, VLSA for short, carries the "whole half" of the structure of vertex superalgebra related to "positive" multiplications, except that properties of **1** in the definition of VSA are replaced by the half skew symmetry in the definition of VLSA. The half skew symmetry can be written by components as

$$u_n v = -\varepsilon_{u,v} \sum_{k \geq 0} (-1)^{n+k} (D^k/k!) v_{n+k} u \quad \text{for all } n \geq 0. \tag{3.8}$$

Also note that in the case when $U = U^0$, i.e., when all vectors are even, all $\mathbb{Z}_2$-grading conditions become trivial and we speak of a vertex Lie algebra, VLA for short.

We define a homomorphism $\varphi: U \to W$ of two VLSA as a $\mathbb{Z}_2$-grading preserving linear map φ such that

$$\varphi(u_n v) = (\varphi(u))_n(\varphi(v)), \qquad \varphi D = D\varphi.$$

Left (resp. right, two-sided) ideals in U are defined as left (resp. right, two-sided) ideals for all multiplications. Note that an one-sided $\mathbb{Z}_2$-graded ideal in VLSA which is invariant for D must be two-sided ideal because of the half skew symmetry. For a subset $S \subset U$ we denote by $\langle S \rangle$ a vertex Lie superalgebra generated by S, i.e., the smallest vertex Lie superalgebra containing the set S.

A partial justification for our terminology might be the following lemma proved in [1] by Borcherds in the case when U is a vertex algebra:

Lemma 3.1. *Let U be a VLSA. Then U/DU is a Lie superalgebra with a commutator* $[u + DU, v + DU] = u_0 v + DU$.

Proof. Since $(Du)_0 = 0$, DU is invariant for right multiplications ${}_0 v$, and then by the skew symmetry (3.8) for left multiplications u_0 as well. Hence on the quotient U/DU we have a well defined bilinear operation $u_0 v$ such that $u_0 v = -\varepsilon_{u,v} v_0 u$. Now for $k=n=m=0$ in (3.2) we get $u_0(v_0w)-\varepsilon_{u,v}v_0(u_0w)=(u_0v)_0w$, a Lie superalgebra Jacobi identity. □

It is clear that any vertex superalgebra V may be viewed as a VLSA. Moreover, any subspace $U\subset V$ invariant for D and closed for multiplications u_nv for $n\geq 0$ is a VLSA. Note that in this case

$$Y_U(u,z)=Y_V^+(u,z)=\sum_{n\geq 0}u_nz^{-n-1}\simeq Y_V(u,z),$$

so that Y for U should not be confused with Y for V.

For a given vertex superalgebra V we shall be especially interested in vertex Lie superalgebras $\langle S\rangle\subset V$ generated by a subspace S and the derivation D, i.e., when vertex Lie superalgebras $\langle S\rangle\subset V$ are of the form

$$\langle S\rangle=\sum_{k\geq 0}D^kS.$$

In such a case any element A in $\langle S\rangle$ can be written as a combination of vectors of the form $P(D)u$, $u\in S$, P a polynomial, and (3.4) implies

$$Y_U(P(D)u,z)Q(D)v=P(\mathrm{d}/\mathrm{d}z)Q(D-\mathrm{d}/\mathrm{d}z)Y_U(u,z)v.$$

Hence the structure of vertex Lie superalgebra $\langle S\rangle\subset V$ is completely determined by $Y_S(u,z)v=Y_U(u,z)v$ for $u,v\in S$. Since

$$\begin{aligned}Y_V(u,z_1)v&=\mathrm{Res}_{z_2}z_2^{-1}Y_V(u,z_1)Y_V(v,z_2)\mathbf{1}\\&\simeq\mathrm{Res}_{z_2}z_2^{-1}(Y_V(u,z_1)Y_V(v,z_2)\mathbf{1}-\varepsilon_{u,v}Y_V(v,z_2)Y_V(u,z_1)\mathbf{1})\\&=\mathrm{Res}_{z_2}z_2^{-1}[Y_V(u,z_1),Y_V(v,z_2)]\mathbf{1},\end{aligned}$$

we may use (for $m\geq 0$)

$$\mathrm{Res}_{z_2}z_2^{-1}z_1^{-m-1}\delta^{(m)}\left(\frac{z_2}{z_1}\right)e^{z_2D}w\simeq m!wz_1^{-m-1},$$

$$\mathrm{Res}_{z_1}z_1^{-1}z_1^{-m-1}\delta^{(m)}\left(\frac{z_2}{z_1}\right)e^{z_2D}w\simeq(-1)^mm!\sum_{k=0}^{m}\frac{1}{k!}D^kwz_1^{-m-1+k},$$

and easily calculate the principal part of the Laurent series $Y_V(u,z)v$ from a given commutator. For example, in the case of Neveu–Schwarz algebra (cf. [12, (4.2.10)])

$$[Y(\omega,z_1),Y(\tau,z_2)]=z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)Y(D\tau,z_2)+\frac{3}{2}z_1^{-2}\delta'\left(\frac{z_2}{z_1}\right)Y(\tau,z_2)$$

gives

$$Y(\omega,z)\tau\simeq\sum_{n\geq 0}\omega_n\tau z^{-n-1}=\frac{D\tau}{z}+\frac{(3/2)\tau}{z^2}.$$

The following examples of VLSA are obtained in this way from the well known vertex (super)algebras associated to affine Lie algebras, Virasoro algebra and Neveu–Schwarz algebra; for details one may see [12, Section 4]:

The case of affine Lie algebras. Here $S = S^0 = \boldsymbol{g} \oplus \mathbb{C}\mathbf{1}$ is a sum of 1-dimensional space and a Lie algebra $\boldsymbol{g}$ with an invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$,

$$\langle S\rangle = \mathbb{C}\mathbf{1} \oplus \operatorname{span}\{D^k x \mid x \in \boldsymbol{g},\ k \geq 0\}$$

and

$$Y_S(x,z)y = \frac{[x,y]}{z} + \frac{\langle x,y\rangle\mathbf{1}}{z^2}, \qquad Y_S(x,z)\mathbf{1} = 0, \qquad Y_S(\mathbf{1},z) = 0$$

for $x, y \in \boldsymbol{g}$.

A case of affine Lie superalgebras. Here $S = \boldsymbol{g} \oplus M \oplus \mathbb{C}\mathbf{1}$ is a sum of 1-dimensional space, a Lie algebra $\boldsymbol{g}$ and a $\boldsymbol{g}$-module M, where both $\boldsymbol{g}$ and M have $\boldsymbol{g}$-invariant symmetric bilinear forms denoted as $\langle\cdot,\cdot\rangle$, with $S^0 = \boldsymbol{g} \oplus \mathbb{C}\mathbf{1}$ and $S^1 = M$,

$$\langle S\rangle = \mathbb{C}\mathbf{1} \oplus \operatorname{span}\{D^k x, D^k u \mid x \in \boldsymbol{g},\ u \in M,\ k \geq 0\}$$

and

$$\begin{aligned}
&Y_S(x,z)y = \frac{[x,y]}{z} + \frac{\langle x,y\rangle\mathbf{1}}{z^2}, \qquad Y_S(x,z)\mathbf{1} = 0,\\
&Y_S(u,z)v = \frac{\langle u,v\rangle\mathbf{1}}{z}, \qquad Y_S(u,z)\mathbf{1} = 0,\\
&Y_S(x,z)u = \frac{x\cdot u}{z}, \qquad Y_S(u,z)x = -\frac{x\cdot u}{z},\\
&Y_S(\mathbf{1},z) = 0
\end{aligned}$$

for $x, y \in \boldsymbol{g}$ and $u, v \in M$.

The case of Virasoro algebra. Here $S = S^0 = \mathbb{C}\omega \oplus \mathbb{C}\mathbf{1}$ is a 2-dimensional space and $\ell \in \mathbb{C}$,

$$\langle S\rangle = \mathbb{C}\mathbf{1} \oplus \operatorname{span}\{D^k \omega \mid k \geq 0\}$$

and

$$Y_S(\omega,z)\omega = \frac{D\omega}{z} + \frac{2\omega}{z^2} + \frac{(\ell/2)\mathbf{1}}{z^4}, \qquad Y_S(\omega,z)\mathbf{1} = 0, \qquad Y_S(\mathbf{1},z) = 0.$$

The case of Neveu–Schwarz algebra. Here $S = S^0 = \mathbb{C}\omega \oplus \mathbb{C}\tau \oplus \mathbb{C}\mathbf{1}$ is a 3-dimensional space and $\ell \in \mathbb{C}$, with $S^0 = \mathbb{C}\omega \oplus \mathbb{C}\mathbf{1}$ and $S^1 = \mathbb{C}\tau$,

$$\langle S\rangle = \mathbb{C}\mathbf{1} \oplus \operatorname{span}\{D^k \omega, D^k \tau \mid k \geq 0\}$$

and

$$Y_S(\omega,z)\omega=\frac{D\omega}{z}+\frac{2\omega}{z^2}+\frac{(\ell/2)\mathbf{1}}{z^4},\qquad Y_S(\omega,z)\mathbf{1}=0,$$

$$Y_S(\tau,z)\tau=\frac{2\omega}{z}+\frac{(2\ell/3)\mathbf{1}}{z^3},\qquad Y_S(\tau,z)\mathbf{1}=0,$$

$$Y_S(\omega,z)\tau=\frac{D\tau}{z}+\frac{(3/2)\tau}{z^2},\qquad Y_S(\tau,z)\omega=\frac{(1/2)D\tau}{z}+\frac{(3/2)\tau}{z^2},$$

$$Y_S(\mathbf{1},z)=0.$$

Remark 3.2. Since for a vertex Lie superalgebra there is no notion of a vacuum vector, it would be better to denote the vector **1** in the above examples by some other letter. So later on we shall set in the affine case $S^0=\boldsymbol{g}\oplus\mathbb{C}c$ with

$$Y_S(x,z)y=\frac{[x,y]}{z}+\frac{\langle x,y\rangle c}{z^2};$$

in the Virasoro case $S^0=\mathbb{C}\omega\oplus\mathbb{C}c$ with

$$Y_S(\omega,z)\omega=\frac{D\omega}{z}+\frac{2\omega}{z^2}+\frac{(1/2)c}{z^4}.$$

In all of these examples the set $S\subset V$ generates the corresponding vertex superalgebra V. In particular, the vertex Lie superalgebra $U=\langle S\rangle\subset V$ generates the corresponding vertex superalgebra V, and obviously this is the smallest such vertex Lie superalgebra. On the other extreme, for a $\frac{1}{2}\mathbb{Z}_+$-graded vertex operator superalgebra V we have $U=\langle S\rangle=\bigoplus_{k\geq 0}D^kS=V$ if we take S to be the set of all quasi-primary fields, i.e., $S=\{u\in V\mid L(1)u=0\}$.

4. Lie algebras associated to vertex Lie algebras

Let U be a vertex Lie superalgebra with the derivation D and consider the affinization $U\otimes\mathbb{C}[q,q^{-1}]$ as a $\mathbb{Z}_2$-graded tensor product of U with the even space $\mathbb{C}[q,q^{-1}]$. For vectors in $U\otimes\mathbb{C}[q,q^{-1}]$ we shall write

$$u_n=u\otimes q^n$$

when $u\in U$ and $n\in\mathbb{Z}$. Consider the quotient $\mathbb{Z}_2$-graded vector space

$$\mathscr{L}(U)=(U\otimes\mathbb{C}[q,q^{-1}])/\mathrm{span}\{(Du)_n+nu_{n-1}\mid u\in U,\ n\in\mathbb{Z}\}.$$

The image of the vector u_n in the quotient space we again denote by u_n. Note that we have three meanings for u_n, originally for $n\geq 0$ as a linear operator on U, now also for all $n\in\mathbb{Z}$ as an element of $U\otimes\mathbb{C}[q,q^{-1}]$ or $\mathscr{L}(U)$, but it will be clear from the context which meaning we have in mind.

Theorem 4.1. *$\mathscr{L}(U)$ is a Lie superalgebra with the commutator defined by*

$$[u_n, v_p] = \sum_{i\geq 0} \binom{n}{i} (u_i v)_{n+p-i} \tag{4.1}$$

for $u, v \in U$ and $n, p \in \mathbb{Z}$. Moreover, we have the relation

$$(Du)_n = -nu_{n-1} \tag{4.2}$$

for all $u \in U$, $n \in \mathbb{Z}$, and the map $D: \mathscr{L}(U) \to \mathscr{L}(U)$ defined by $D(u_n) = (Du)_n$ is an even derivation of the Lie superalgebra $\mathscr{L}(U)$.

In the case when U is a vertex algebra this theorem is a special case of Borcherds' Lemma 3.1 applied to the affinization of U, in such a case a tensor product of vertex algebras. Direct proofs are also known. See [11, Proposition 2.2.3, Remark 2.2.4]. Here we follow [16] where the techniques introduced in [7] are used. By following [4] we may call $\mathscr{L}(U)$ the local algebra of the vertex Lie algebra U. This notion of local algebra $\mathscr{L}(U)$ is a special case of a more general notion of local vertex Lie algebra over the base space U introduced in [3].

It will be convenient to consider formal Laurent series (vertex operators)

$$Y(u,z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1} \tag{4.3}$$

with coefficients in $\mathscr{L}(U)$. Then we can write relations (4.1) and (4.2) as

$$[Y(u,z_1), Y(v,z_2)] = \mathrm{Res}_{z_0} z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v, z_2), \tag{4.4}$$

$$Y(Du,z) = \frac{\mathrm{d}}{\mathrm{d}z} Y(u,z). \tag{4.5}$$

Note that in (4.4) there are three Y's with coefficients in $\mathscr{L}(U)$ and $Y(u,z_0)$ defined as $\sum_{n\geq 0} u_n z_0^{-n-1}$ has coefficients in $\mathrm{End}(U)$. Also note that in the expansion of δ-function there are only nonnegative powers of z_0, so when we calculate the residue Res_{z_0} we need to know only the principal part of $Y(u,z_0)$. For this reason the commutator formula (4.4) written in components looks the same as in the vertex algebra case. This is the key observation used in the proof of the above theorem.

Proof. First note that for vectors $u_n = u \otimes q^n$ formula (4.1) defines a bilinear operation $[\cdot,\cdot]$ which makes $U \otimes \mathbb{C}[q,q^{-1}]$ a $\mathbb{Z}_2$-graded algebra. Since D is an even derivation of U, it is obvious from (4.1) that $D = D \otimes 1$ is an even derivation of $[\cdot,\cdot]$. Relations (3.4) for the derivation D of VLSA U imply

$$[(Du)_n + nu_{n-1}, v_p] = 0, \qquad [u_n, (Dv)_p + pv_{p-1}] \equiv 0.$$

Here the first relation follows from $(Du)_i = -iu_{i-1}$, for the second we also need $u_i(Dv) = D(u_i v) - (Du)_i v$ and the result is in $\mathrm{span}\{(Dw)_n + nw_{n-1}\}$. Hence the bilinear operation on the quotient space $\mathscr{L}(U)$ is well defined. It is clear $\mathscr{L}(U)$ is

$\mathbb{Z}_2$-graded algebra and that D is well defined even derivation such that (4.2) holds. It remains to check the axioms of Lie superalgebra. Since

$$[Y(u,z_1),Y(v,z_2)] = \mathrm{Res}_{z_0} z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y(Y(u,z_0)v,z_2) \tag{4.6}$$

$$= \varepsilon_{u,v}\,\mathrm{Res}_{z_0} z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y(\mathrm{e}^{z_0 D}Y(v,-z_0)u,z_2) \tag{4.7}$$

$$= \varepsilon_{u,v}\,\mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right) Y(Y(v,-z_0)u,z_2+z_0) \tag{4.8}$$

$$= \varepsilon_{u,v}\,\mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right) Y(Y(v,-z_0)u,z_1)$$

$$= -\varepsilon_{u,v}[Y(v,z_2),Y(u,z_1)],$$

the skew symmetry for Lie superalgebra holds. Note that (4.7) follows from (4.6) by using the half skew symmetry (3.5). As it was already mentioned, for getting this equality it was sufficient to know the equality for principal parts of Laurent series in variable z_0. Note that (4.8) follows from (4.7) because (4.5) holds.

To prove the Jacobi identity for Lie superalgebras we use the definition of commutator and get

$$[[Y(u,z_1),Y(v,z_2)],Y(w,z_3)]$$

$$= \mathrm{Res}_{z_{23}} z_2^{-1}\delta\left(\frac{z_3+z_{23}}{z_2}\right)\mathrm{Res}_{z_{12}} z_1^{-1}\delta\left(\frac{z_2+z_{12}}{z_1}\right) Y(Y(Y(u,z_{12})v,z_{23})w,z_3).$$

Note that in the expansions of δ-functions there are only the nonnegative powers of z_{12} and z_{23} and that, by taking the residues, the above expression involves only the coefficients of the principal part of $Y(Y(u,z_{12})v,z_{23})w$. Hence we may apply the half associator formula (6.8) and get

$$[[Y(u,z_1),Y(v,z_2)],Y(w,z_3)]$$

$$= \mathrm{Res}_{z_{23}} z_2^{-1}\delta\left(\frac{z_3+z_{23}}{z_2}\right)\mathrm{Res}_{z_{12}} z_1^{-1}\delta\left(\frac{z_2+z_{12}}{z_1}\right)\mathrm{Res}_{z_{13}} z_{12}^{-1}\delta\left(\frac{z_{13}-z_{23}}{z_{12}}\right)$$

$$\times\, Y(Y(u,z_{13})Y(v,z_{23})w,z_3) \tag{4.9}$$

$$-\varepsilon_{u,v}\,\mathrm{Res}_{z_{23}} z_2^{-1}\delta\left(\frac{z_3+z_{23}}{z_2}\right)\mathrm{Res}_{z_{12}} z_1^{-1}\delta\left(\frac{z_2+z_{12}}{z_1}\right)\mathrm{Res}_{z_{13}} z_{12}^{-1}\delta\left(\frac{z_{23}-z_{13}}{-z_{12}}\right)$$

$$\times\, Y(Y(v,z_{23})Y(u,z_{13})w,z_3) \tag{4.10}$$

$$= \mathrm{Res}_{z_{13}} \mathrm{Res}_{z_{23}} z_1^{-1}\delta\left(\frac{z_3+z_{13}}{z_1}\right) z_2^{-1}\delta\left(\frac{z_3+z_{23}}{z_2}\right)$$

$$\times Y(Y(u,z_{13})Y(v,z_{23})w,z_3) \tag{4.11}$$

$$-\varepsilon_{u,v} \mathrm{Res}_{z_{23}} \mathrm{Res}_{z_{13}} z_1^{-1}\delta\left(\frac{z_3+z_{13}}{z_1}\right) z_2^{-1}\delta\left(\frac{z_3+z_{23}}{z_2}\right)$$

$$\times Y(Y(v,z_{23})Y(u,z_{13})w,z_3) \tag{4.12}$$

$$= [Y(u,z_1),[Y(v,z_2),Y(w,z_3)]] - \varepsilon_{u,v}[Y(v,z_2),[Y(u,z_1),Y(w,z_3)]],$$

which is the Jacobi identity for Lie superalgebra $\mathscr{L}(U)$. Note that we obtained the expression (4.12) from (4.10) by using the identity for δ-functions:

$$\mathrm{Res}_{z_{12}} z_2^{-1}\delta\left(\frac{z_3+z_{23}}{z_2}\right) z_1^{-1}\delta\left(\frac{z_2+z_{12}}{z_1}\right) z_{12}^{-1}\delta\left(\frac{z_{13}-z_{23}}{z_{12}}\right)$$

$$= \mathrm{Res}_{z_{12}} z_2^{-1}\delta\left(\frac{z_3+z_{23}}{z_2}\right) z_{12}^{-1}\delta\left(\frac{z_{13}-z_{23}}{z_{12}}\right) \sum_n \sum_{r\geq 0} \binom{n}{r} z_2^{n-r} z_{12}^r z_1^{-n-1}$$

$$= \mathrm{Res}_{z_{12}} z_2^{-1}\delta\left(\frac{z_3+z_{23}}{z_2}\right) z_{12}^{-1}\delta\left(\frac{z_{13}-z_{23}}{z_{12}}\right)$$

$$\times \sum_n \sum_{r\geq 0} \binom{n}{r} (z_3+z_{23})^{n-r}(z_{13}-z_{23})^r z_1^{-n-1}$$

$$= z_2^{-1}\delta\left(\frac{z_3+z_{23}}{z_2}\right) \sum_n \sum_{r\geq 0} \sum_{s\geq 0} \binom{n}{r}\binom{n-r}{s} z_3^{n-r-s} z_{23}^s (z_{13}-z_{23})^r z_1^{-n-1}$$

$$= z_2^{-1}\delta\left(\frac{z_3+z_{23}}{z_2}\right) \sum_n \sum_{k\geq 0} \sum_{s+r=k} \binom{n}{k}\binom{k}{r} z_3^{n-k} z_{23}^s (z_{13}-z_{23})^r z_1^{-n-1}$$

$$= z_2^{-1}\delta\left(\frac{z_3+z_{23}}{z_2}\right) \sum_n \sum_{k\geq 0} \binom{n}{k} z_3^{n-k} z_{13}^k z_1^{-n-1}$$

$$= z_2^{-1}\delta\left(\frac{z_3+z_{23}}{z_2}\right) z_1^{-1}\delta\left(\frac{z_3+z_{13}}{z_1}\right).$$

This identity for δ-functions should be understood in the context of the above proof, i.e., in the presence of the term $Y(Y(u,z_{13})Y(v,z_{23})w,z_3)$ and the residues $\mathrm{Res}_{z_{13}} \mathrm{Res}_{z_{23}}$. We obtain the expression (4.13) from (4.11) in a similar way. □

Remark 4.2. Motivated by a definition of local vertex Lie algebra over the base space U given in [3] and by methods used in [9], let us note the following:

Given a vector space U and a linear map Y satisfying (3.3), (3.4) and (3.7), later on we shall say that U is a VLA-J-SS, we have on the quotient space $\mathscr{L}(U)$ a bilinear operation $[\cdot,\cdot]$ defined by (4.1). Then $\mathscr{L}(U)$ is a Lie (super)algebra if and only if U is a vertex Lie (super)algebra.

To see the "only if" part, note that, by Theorem 4.6 below, $u=0$ if and only if $Y(u,z)=0$. The skew symmetry for $[\cdot,\cdot]$ implies that $(4.6)=(4.7)$ and, by the above remark and Lemma 2.1.4 in [12], this implies the half skew symmetry (3.5). In a similar way we see that the Jacobi identity for $[\cdot,\cdot]$ implies the half associator formula (6.8), which by Lemma 6.2 below implies the half Jacobi identity (3.6).

Proposition 4.3. *Let $\varphi : U\to W$ be a homomorphism of vertex Lie superalgebras. Then*

$$\mathscr{L}(\varphi):\mathscr{L}(U)\to\mathscr{L}(W),\qquad u_n\mapsto(\varphi(u))_n \tag{4.13}$$

is a homomorphism of Lie superalgebras. Moreover, $\mathscr{L}(\varphi)D=D\mathscr{L}(\varphi)$.

Proof. Since $\varphi D=D\varphi$, the map $u\otimes q^n\to\varphi(u)\otimes q^n$ factors through the quotient and the map $\mathscr{L}(\varphi)$ is well defined $\mathbb{Z}_2$-grading preserving map such that $\mathscr{L}(\varphi)D=D\mathscr{L}(\varphi)$. From the definition of commutator (4.1) we have

$$\sum_{i\ge 0}\binom{n}{i}(u_iv)_{n+p-i}\mapsto\sum_{i\ge 0}\binom{n}{i}(\varphi(u_iv))_{n+p-i}=\sum_{i\ge 0}\binom{n}{i}(\varphi(u)_i\varphi(v))_{n+p-i},$$

i.e., $\mathscr{L}(\varphi)$ is a homomorphism of Lie superalgebras. □

Set $\mathscr{L}_-(U)=\operatorname{span}\{u_n\mid u\in U,\ n<0\}$, $\mathscr{L}_+(U)=\operatorname{span}\{u_n\mid u\in U,\ n\ge 0\}$.

Proposition 4.4. *$\mathscr{L}_-(U)$ and $\mathscr{L}_+(U)$ are Lie superalgebras invariant for D and*

$$\mathscr{L}(U)=\mathscr{L}_-(U)\oplus\mathscr{L}_+(U),$$

$$\mathscr{L}_-(U)\cong(U\otimes q^{-1}\mathbb{C}[q^{-1}])/\operatorname{span}\{(Du)_n+nu_{n-1}\mid u\in U,\ n<0\},$$

$$\mathscr{L}_+(U)\cong(U\otimes\mathbb{C}[q])/\operatorname{span}\{(Du)_n+nu_{n-1}\mid u\in U,\ n\ge 0\}.$$

Proof. It is clear from definition (4.1) of commutator in $\mathscr{L}(U)$ that both $\mathscr{L}_-(U)$ and $\mathscr{L}_+(U)$ are subalgebras. The invariance for D follows from (4.2). The other statements follow from the fact that $\mathscr{L}(U)$ is the quotient of

$$(U\otimes q^{-1}\mathbb{C}[q^{-1}])\oplus(U\otimes\mathbb{C}[q])$$

by the sum of $\mathbb{Z}_2$-graded D invariant subspaces

$$\operatorname{span}\{(Du)_n+nu_{n-1}\mid u\in U,\ n<0\}\oplus\operatorname{span}\{(Du)_n+nu_{n-1}\mid u\in U,\ n\ge 0\}.\quad\square$$

The following proposition is an obvious consequence of definition (4.13) of $\mathscr{L}(\varphi)$:

Proposition 4.5. *Let $\varphi : U \to W$ be a homomorphism of vertex Lie superalgebras and let $\mathscr{L}_\pm(\varphi)$ be the restriction of $\mathscr{L}(\varphi)$ on $\mathscr{L}_\pm(U)$. Then $\mathscr{L}_\pm(\varphi)(\mathscr{L}_\pm(U)) \subset \mathscr{L}_\pm(W)$ and*

$$\mathscr{L}_-(\varphi) : \mathscr{L}_-(U) \to \mathscr{L}_-(W), \qquad \mathscr{L}_+(\varphi) : \mathscr{L}_+(U) \to \mathscr{L}_+(W)$$

are homomorphisms of Lie superalgebras. Moreover, $\mathscr{L}_\pm(\varphi)D = D\mathscr{L}_\pm(\varphi)$.

Theorem 4.6. *The map*

$$\iota_U : U \to \mathscr{L}_-(U), \quad u \mapsto u_{-1}$$

is an isomorphism of $\mathbb{Z}_2$-graded vector spaces U and $\mathscr{L}_-(U)$ and $\iota_U D = D\iota_U$.

Moreover, if $\varphi : U \to W$ is a homomorphism of vertex Lie superalgebras, then $\iota_W \varphi = \mathscr{L}_-(\varphi)\iota_U$.

Proof. It follows from (4.2) that $u_{-k-1} = (1/k!)(D^k u)_{-1}$ for $k \geq 0$, and this implies that ι_U is surjective. Let $u_{-1} = 0$, that is

$$u \otimes q^{-1} = \sum_{i=-n}^{m} (Dv^{(i)} \otimes q^i + iv^{(i)} \otimes q^{i-1})$$

for some $n, m \geq 1$ and some $v^{(i)} \in U$. Obviously $\sum_{i \geq 0} (Dv^{(i)} \otimes q^i + iv^{(i)} \otimes q^{i-1}) = 0$. Since $v^{(i)} = 0$ implies $Dv^{(i)} = 0$, by induction we get $v^{(i)} = 0$ for all $i < 0$, and hence $u = 0$. So ι_U is an isomorphism. The remaining statements are clear. □

We may identify U with $\mathscr{L}_-(U)$ via the map ι_U and consider

$$U \subset \mathscr{L}(U) = U \oplus \mathscr{L}_+(U).$$

If we transport the Lie superalgebra structure of $\mathscr{L}_-(U)$ on U, then commutator (4.1) reads for $u, v \in U$ as

$$[u, v] = \sum_{n \geq 0} ((-1)^n/(n+1)!) D^{n+1}(u_n v). \tag{4.14}$$

Note that by Theorem 4.6 any homomorphism $\varphi : U \to W$ of vertex Lie superalgebras is a homomorphism of Lie superalgebras U and W with commutators defined by (4.14). Moreover, this homomorphism φ extends to a homomorphism of Lie superalgebras $\mathscr{L}(U)$ and $\mathscr{L}(W)$.

5. Enveloping vertex algebras of vertex Lie algebras

Let U be a vertex Lie superalgebra with the derivation D and let $\mathscr{L}(U) = \mathscr{L}_-(U) \oplus \mathscr{L}_+(U)$ be the corresponding Lie superalgebra with the derivation D. The induction by a trivial $\mathscr{L}_+(U)$-module $\mathbb{C}$ gives a generalized Verma $\mathscr{L}(U)$-module

$$\mathscr{V}(U) = \mathscr{U}(\mathscr{L}(U)) \otimes_{\mathscr{U}(\mathscr{L}_+(U))} \mathbb{C},$$

where $\mathscr{U}$ stands for the universal enveloping algebra of a given Lie superalgebra. The $\mathscr{L}(U)$-module $\mathscr{V}(U)$ is isomorphic to a quotient

$$\mathscr{V}(U)=\mathscr{U}(\mathscr{L}(U))/\mathscr{U}(\mathscr{L}(U))\mathscr{L}_+(U)$$

of $\mathscr{U}(\mathscr{L}(U))$ by a left ideal generated by $\mathscr{L}_+(U)$. Clearly $\mathscr{V}(U)$ is a $\mathbb{Z}_2$-graded space and the action of Lie superalgebra $\mathscr{L}(U)$ is given by the left multiplication. Note that the derivation D of Lie superalgebra $\mathscr{L}(U)$ extends to a derivation D of the associative superalgebra $\mathscr{U}(\mathscr{L}(U))$, and since D preserves $\mathscr{L}_+(U)$, if defines an even operator D on the quotient $\mathscr{V}(U)$. We denote by $\mathbf{1}\in\mathscr{V}(U)$ the image of $1\in\mathscr{U}(\mathscr{L}(U))$. It is an even vector and we have

$$D\mathbf{1}=0.$$

If we think of $\mathscr{V}(U)$ only as a $\mathbb{Z}_2$-graded space, then

$$\mathscr{V}(U)\cong\mathscr{U}(\mathscr{L}(U)_-)$$

and under this identification $\mathbf{1}$ is the identity and D is an even derivation of the associative superalgebra $\mathscr{U}(\mathscr{L}_-(U))$ extending the derivation D of $\mathscr{L}_-(U)$. By Theorem 4.6 we may identify U and $\mathscr{L}_-(U)$ via the map $\imath_U$, so clearly the map

$$\kappa_U : U\to\mathscr{V}(U),\quad u\mapsto u_{-1}\mathbf{1}$$

is an injection. Sometimes it will be convenient to identify U and the $\mathbb{Z}_2$-graded subspace $\kappa_U(U)\subset\mathscr{V}(U)$, i.e., to consider

$$U\subset\mathscr{V}(U). \tag{5.1}$$

Lemma 5.1. *For the action of $u_n\in\mathscr{L}(U)$ on the module $\mathscr{V}(U)$ we have*

$$[D,u_n]=(Du)_n=-nu_{n-1}. \tag{5.2}$$

Proof. By definition the operator D acts on a vector

$$w=u^{(1)}\dots u^{(k)}\mathbf{1}\in\mathscr{V}(U),\qquad u^{(1)},\dots,u^{(k)}\in\mathscr{L}(U), \tag{5.3}$$

as a derivation, and by definition u_n acts by the left multiplication, so $Du_nw-u_nDw=D(u_n)w$. Hence $[D,u_n]=D(u_n)=(Du)_n$ and relation (4.2) implies the lemma. □

Since $D\mathbf{1}=0$, we have $D(u_{-1}\mathbf{1})=[D,u_{-1}]\mathbf{1}=(Du)_{-1}\mathbf{1}$. Hence $D\kappa_U=\kappa_UD$ and on $U\subset\mathscr{V}(U)$ both derivations coincide.

Since $\mathscr{V}(U)$ is a $\mathscr{L}(U)$-module, the formal Laurent series $Y(u,z)$ defined by (4.3) operate on $\mathscr{V}(U)$ and the corresponding formal Laurent series we shall denote by $Y_{\mathscr{V}(U)}(u,z)$. Hence the coefficients u_n in

$$Y_{\mathscr{V}(U)}(u,z)=Y_{\mathscr{V}(U)}(u_{-1}\mathbf{1},z)=\sum_{n\in\mathbb{Z}}u_nz^{-n-1},\quad u\in U$$

are operators on $\mathscr{V}(U)$. Then we can restate (5.2) as

$$[D, Y_{\mathscr{V}(U)}(u,z)] = Y_{\mathscr{V}(U)}(Du,z) = \frac{\mathrm{d}}{\mathrm{d}z} Y_{\mathscr{V}(U)}(u,z). \tag{5.4}$$

Lemma 5.2. *The set of formal Laurent series* $\{Y_{\mathscr{V}(U)}(u,z) \mid u \in U^0 \cup U^1\}$ *is a set of mutually local vertex operators on* $\mathscr{V}(U)$.

Proof. We have already proved (5.4), that is (2.14) in the definition of vertex operators. By construction we have

$$u_n \mathbf{1} = 0 \quad \text{for all } u \in U, \ n \geq 0. \tag{5.5}$$

Hence for a vector $w \in \mathscr{V}(U)$ of the form (5.3) we have

$$u_n w = u_n u^{(1)} \ldots u^{(k)} \mathbf{1} = [u_n, u^{(1)} \ldots u^{(k)}] \mathbf{1}$$

for $n \geq 0$, and then the commutator formula (4.1) implies $u_n w = 0$ for sufficiently large n, i.e., (2.7) in the definition of vertex operators. The commutator formula (4.4) for $u, v \in U^0 \cup U^1$ written in the form (3.1) clearly implies locality and the lemma follows. □

Since (5.5) means that $Y_{\mathscr{V}(U)}(u,z)\mathbf{1}$ is a power series in z and since by construction $\lim_{z \to 0} Y_{\mathscr{V}(U)}(u,z)\mathbf{1} = u_{-1}\mathbf{1} = u$, all the assumptions of Theorem 2.2 hold and we have the following:

Theorem 5.3. *The set* $\{Y_{\mathscr{V}(U)}(u,z) \mid u \in U^0 \cup U^1\}$ *of mutually local vertex operators on* $\mathscr{V}(U)$ *generates the vertex superalgebra structure on* $\mathscr{V}(U)$ *with the vacuum vector* $\mathbf{1}$ *and the derivation* D.

Proposition 5.4. *The map* $\kappa_U : U \to \mathscr{V}(U)$ *is an injective homomorphism of vertex Lie superalgebras.*

Proof. We have already seen that κ_U is an injective $\mathbb{Z}_2$-grading preserving map such that $\kappa_U D = D \kappa_U$. So let $n \geq 0$. Because of (5.5) and the commutator formula (4.1) we have

$$u_n(v_{-1}\mathbf{1}) = [u_n, v_{-1}]\mathbf{1} = \sum_{i \geq 0} \binom{n}{i} (u_i v)_{n-1-i} \mathbf{1} = \binom{n}{n} (u_n v)_{-1} \mathbf{1}.$$

Hence

$$\kappa_U(u_n v) = (u_n v)_{-1}\mathbf{1} = u_n(v_{-1}\mathbf{1}) = (u_{-1}\mathbf{1})_n (v_{-1}\mathbf{1}) = (\kappa_U(u))_n (\kappa_U(v)). \qquad \square$$

The above proposition means that, with identification (5.1), the structure of vertex superalgebra on $\mathscr{V}(U)$ extends the structure of vertex Lie superalgebra on $U \subset \mathscr{V}(U)$.

As suggested by the next theorem, we shall say that $\mathscr{V}(U)$ is the universal enveloping vertex superalgebra of the vertex Lie superalgebra U.

Theorem 5.5. *Let V be a vertex superalgebra and $\varphi: U \to V$ a homomorphism of vertex Lie superalgebras. Then φ extends uniquely to a vertex superalgebra homomorphism $\tilde{\varphi}: \mathscr{V}(U) \to V$.*

Proof. Set $M = \mathscr{V}(U) \oplus V$ and for $u \in U$ define

$$Y_M(u,z) = Y_{\mathscr{V}(U)}(u,z) + Y_V(\varphi(u),z).$$

Then $\{Y_M(u,z) \mid u \in U^0 \cup U^1\}$ is a set of mutually local vertex operators on M which, by Li's Theorem 2.1, generates a vertex algebra W. Consider the restriction maps

$$\mathscr{V}(U) \xleftarrow{p_1} W \xrightarrow{p_2} V$$

defined by

$$p_1: a(z) \mapsto a(z) \mid \mathscr{V}(U) \mapsto \lim_{z \to 0}(a(z) \mid \mathscr{V}(U))\mathbf{1},$$
$$p_2: a(z) \mapsto a(z) \mid V \mapsto \lim_{z \to 0}(a(z) \mid V)\mathbf{1}.$$

Step 1: *The maps p_1 and p_2 are homomorphisms of vertex superalgebras.* Recall that for two formal Laurent series $u(z), v(z) \in F(M)$ the product $u(z)_n v(z)$ is defined by (2.12), so it is clear that for any invariant subspace $N \subset M$ we have

$$(u(z)_n v(z)) \mid N = (u(z) \mid N)_n (v(z) \mid N).$$

For this reason both $a(z) \mapsto a(z) \mid \mathscr{V}(U)$ and $a(z) \mapsto a(z) \mid V$ are homomorphisms of vertex superalgebras, the first restriction from W to a vertex superalgebra W_1 of vertex operators on $\mathscr{V}(U)$ generated by $(Y_M(u,z)) \mid \mathscr{V}(U) = Y_{\mathscr{V}(U)}(u,z)$, $u \in U$, the second restriction from W to a vertex superalgebra W_2 of vertex operators on V generated by $(Y_M(u,z)) \mid V = Y_V(\varphi(u),z)$, $u \in U$. Since vertex superalgebras are closed for multiplications, we obviously have

$$W_1 \subset \{Y_{\mathscr{V}(U)}(v,z) \mid v \in \mathscr{V}(U)\}, \qquad W_2 \subset \{Y_V(v,z) \mid v \in V\}.$$

Since in general for any vertex superalgebra the map

$$Y(v,z) \mapsto v_{-1}\mathbf{1} = \lim_{z \to 0} Y(v,z)\mathbf{1}$$

is an isomorphism of vertex superalgebra of fields with the algebra itself, both p_1 and p_2 are homomorphisms of vertex superalgebras.

Let us note at this point that p_1 is a surjection since $\mathscr{V}(U) \cong \{Y_{\mathscr{V}(U)}(v,z) \mid v \in \mathscr{V}(U)\}$ is generated by $\{Y_{\mathscr{V}(U)}(u,z) \mid u \in U\}$.

Step 2: *W is a $\mathscr{L}(U)$-module and $W = \mathscr{U}(\mathscr{L}_-(U))I(z)$.* A linear map

$$U \otimes \mathbb{C}[q,q^{-1}] \to \operatorname{End} W, \qquad u \otimes q^n \mapsto u(z)_n = Y_M(u,z)_n$$

is obviously well defined. Since

$$
\begin{aligned}
Du\otimes q^n + nu\otimes q^{n-1} &\mapsto (Du)(z)_n + nu(z)_{n-1}\\
&= Y_{\mathscr{V}(U)}(Du,z)_n + Y_V(\varphi(Du),z)_n\\
&\quad + nY_{\mathscr{V}(U)}(u,z)_{n-1} + nY_V(\varphi(u),z)_{n-1}\\
&= \left(\frac{\mathrm{d}}{\mathrm{d}z}Y_{\mathscr{V}(U)}(u,z)\right)_n + \left(\frac{\mathrm{d}}{\mathrm{d}z}Y_V(\varphi(u),z)\right)_n\\
&\quad + nY_{\mathscr{V}(U)}(u,z)_{n-1} + nY_V(\varphi(u),z)_{n-1}\\
&= \left(\frac{\mathrm{d}}{\mathrm{d}z}u(z)\right)_n + nu(z)_{n-1} = 0,
\end{aligned}
$$

we have a well defined map on the quotient

$$\mathscr{L}(U)\to \operatorname{End} W, \qquad u_n \mapsto u(z)_n = Y_M(u,z)_n.$$

Since $\varphi : U\to V$ is VLA homomorphism, we have for $n\geq 0$

$$
\begin{aligned}
(u_nv)(z) &= Y_{\mathscr{V}(U)}(u_nv,z) + Y_V(\varphi(u_nv),z)\\
&= Y_{\mathscr{V}(U)}(u_nv,z) + Y_V(\varphi(u)_n\varphi(v),z)\\
&= Y_{\mathscr{V}(U)}(u,z)_nY_{\mathscr{V}(U)}(v,z) + Y_V(\varphi(u),z)_nY_V(\varphi(v),z)\\
&= u(z)_nv(z),
\end{aligned}
$$

which together with (4.1) and (3.1) implies

$$
\begin{aligned}
[u_p,v_q] = \sum_{n\geq 0}\binom{p}{n}(u_nv)_{p+q-n} &\mapsto \sum_{n\geq 0}\binom{p}{n}(u_nv)_{p+q-n}\\
&= \sum_{n\geq 0}\binom{p}{n}(u(z)_nv(z))_{p+q-n}\\
&= [u(z)_p, v(z)_q].
\end{aligned}
$$

Hence the map $u_n\mapsto u(z)_n$ is a representation of $\mathscr{L}(U)$ on W.

By definition W is generated by $\mathbf{1}=I(z)$ and the set of homogeneous fields $u(z)=Y_M(u,z)$, $u\in U^0\cup U^1$. This means that we consider all possible products like

$$((u^{(1)}(z)_{n_1}u^{(2)}(z))_{n_2}(u^{(3)}(z)_{n_3}(u^{(4)}(z)_{n_4}u^{(5)}(z))).$$

It is easy to see by using the associator formula (cf. (3.2)) that W is spanned by elements of the form

$$
\begin{aligned}
&u^{(1)}(z)_{n_1}(u^{(2)}(z)_{n_2}(\ldots(u^{(k)}(z)_{n_k}u^{(k+1)}(z))\ldots))\\
&\quad = u^{(1)}(z)_{n_1}u^{(2)}(z)_{n_2}\ldots u^{(k)}(z)_{n_k}u^{(k+1)}(z)_{-1}I(z), \qquad (5.6)
\end{aligned}
$$

$u^{(1)}, u^{(2)}, \ldots, u^{(k+1)} \in U$, $n_1, n_2, \ldots, n_k \in \mathbb{Z}$. Hence $W = \mathscr{U}(\mathscr{L}(U))I(z)$. Since we have $u(z)_n I(z) = 0$ for $n \geq 0$, by using the PBW theorem we finally get $W = \mathscr{U}(\mathscr{L}_-(U))I(z)$.

Step 3: *The map* $p_1 : W \to \mathscr{V}(U)$ *is an isomorphism of vertex superalgebras.* We already know that p_1 is a surjective homomorphism. If we are to identify $v \leftrightarrow Y_{\mathscr{V}(U)}(v,z)$ for $v \in \mathscr{V}(U)$, the map p_1 is just a restriction $a(z) \mapsto a(z) \mid \mathscr{V}(U)$, so on monomials of the form (5.6) we have with our identification

$$p_1 : u^{(1)}(z)_{n_1} u^{(2)}(z)_{n_2} \ldots u^{(k)}(z)_{n_k} u^{(k+1)}(z)_{n_{k+1}} I(z) \mapsto u^{(1)}_{n_1} u^{(2)}_{n_2} \ldots u^{(k)}_{n_k} u^{(k+1)}_{n_{k+1}} \mathbf{1}.$$

Hence for $u \in \mathscr{U}(\mathscr{L}_-(U))$ we have $p_1(uI(z)) = u\mathbf{1} \cong u \in \mathscr{U}(\mathscr{L}_-(U))$. This implies that p_1 is an injection as well and that we have a homomorphism of vertex superalgebras

$$\mathscr{V}(U) \xrightarrow{p_1^{-1}} W \xrightarrow{p_2} V.$$

Set $\tilde{\varphi} = p_2 \circ p_1^{-1}$. Then by construction we have for $u \in U$

$$u \mapsto Y_M(u,z) \mapsto Y_V(\varphi(u),z) \leftrightarrow \varphi(u).$$

Hence $\tilde{\varphi} \mid U = \varphi$. Since U generates the vertex superalgebra $\mathscr{V}(U)$, the homomorphism $\tilde{\varphi}$ is uniquely determined by φ. □

Corollary 5.6. *Let* $\varphi : U_1 \to U_2$ *be a homomorphism of vertex Lie superalgebras* U_1 *and* U_2. *Then* φ *extends uniquely to a vertex superalgebra homomorphism* $\mathscr{V}(\varphi)$: $\mathscr{V}(U_1) \to \mathscr{V}(U_2)$.

We shall say that a $\mathscr{L}(U)$-module M is restricted if for any $u \in U$, $w \in M$

$$u_n w = 0 \quad \text{for } n \text{ sufficiently large.}$$

Let us denote by $\mathscr{L}(U) \times \mathbb{C}D$ a Lie superalgebra with $[D, u_n] = D(u_n)$ and let us say that a $(\mathscr{L}(U) \times \mathbb{C}D)$-module M is restricted if M is restricted as an $\mathscr{L}(U)$-module. Note that $\mathscr{V}(U)$ is a restricted $(\mathscr{L}(U) \times \mathbb{C}D)$-module.

If M is a $\mathscr{L}(U)$-module, then for any $u \in U$ we can form a formal Laurent series $Y_M(u,z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$ with elements $u_n \in \mathscr{L}(U)$ acting as operators on the module M. If M is a restricted $(\mathscr{L}(U) \times \mathbb{C}D)$-module, then $\{Y_M(u,z) \mid u \in U^0 \cup U^1\}$ is a set of mutually local vertex operators on M.

Lemma 5.7. *Let* M *be a restricted* $(\mathscr{L}(U) \times \mathbb{C}D)$*-module and let* W *be a vertex superalgebra generated by* $\{Y_M(u,z) \mid u \in U^0 \cup U^1\}$. *Set* $u(z) = Y_M(u,z)$ *and denote by* $u(z)_n$ *operators on* W *defined by multiplications* (2.12). *Then the linear map*

$$\mathscr{L}(U) \to \operatorname{End} W, \quad u_n \mapsto u(z)_n$$

is well defined and

$$(u_n v)(z) = u(z)_n v(z) \quad \textit{for all } n \geq 0,$$

$$[u_n, v_m] \mapsto [u(z)_n, v(z)_m] \quad \textit{for all } n, m \in \mathbb{Z}.$$

In another words, the map

$$U \to W, \qquad u \mapsto u(z) \tag{5.7}$$

is a homomorphism of vertex Lie superalgebras and W is a restricted $(\mathscr{L}(U) \times \mathbb{C}D)$-module.

Proof. Set $N = \mathscr{V}(U) \oplus M$ and for $u \in U$

$$Y_N(u,z) = Y_{\mathscr{V}(U)}(u,z) + Y_M(u,z).$$

This vertex operators on N generate a vertex superalgebra V, denote by p_1 a restriction map from V to $\mathscr{V}(U)$ and by p_2 a restriction map from V to W. Then both p_1 and p_2 are homomorphisms of vertex superalgebras and both $\mathscr{V}(U)$ and W are V-modules. By [12, Lemma 2.3.5] the "structure constants" in the commutator formula for V are completely determined by the "structure constants" appearing in the commutator formula for a faithful V-module. Since for $u(z) \in V$ we have $Y_{\mathscr{V}(U)}(u(z), z_1) = Y_{\mathscr{V}(U)}(u, z_1)$, i.e., $u(z)_n$ acts on $\mathscr{V}(U)$ as u_n, the commutator formula for V-module $\mathscr{V}(U)$ reads

$$[u(z)_p, v(z)_q] = [u_p, v_q] = \sum_{n \geq 0} \binom{p}{n} (u_n v)(z)_{p+q-n}$$

and implies the commutator formula for V-module M

$$[u(z)_p, v(z)_q] = \sum_{n \geq 0} \binom{p}{n} (u_n v)(z)_{p+q-n}.$$

Since $[u(z)_p, v(z)_q]$ equals $\sum_{n \geq 0} \binom{p}{n} (u(z)_n v(z))_{p+q-n}$, the lemma follows. □

Since by Theorem 5.5 a VLSA homomorphism (5.7) extends to a VSA homomorphism $\mathscr{V}(U) \to W$, and since by Li's Theorem 2.1 M is a W-module, we have the following:

Theorem 5.8. *Any restricted $(\mathscr{L}(U) \times \mathbb{C}D)$-module is a $\mathscr{V}(U)$-module.*

Remark 5.9. The proofs of Theorems 5.5, 5.8 and Lemma 5.7 are modeled after the arguments in [12, Section 4]. Also note that Theorem 5.3 can be proved without using Theorem 2.2, but rather by using the proof of Theorem 5.5 with Step 2 changed with the help of Lemma 5.7. This would be parallel to the argument in [12] when proving that for an affine Lie algebra $\tilde{g}$ the generalized Verma module $M_g(\ell, \mathbb{C})$ is a vertex algebra and that any restricted $\tilde{g}$-module of level ℓ is a $M_g(\ell, \mathbb{C})$-module.

The results in this section are similar to some results in [3].

6. Commutativity and the skew symmetry for VLA

For the purposes of the last two sections let us make a few technical definitions: Let U be a $\mathbb{Z}_2$-graded vector space equipped with an even linear operator D on U called the derivation and a linear map $U \to z^{-1}(\mathrm{End}\, U)[[z^{-1}]]$, $u \mapsto Y(u,z) = \sum_{n \geq 0} u_n z^{-n-1}$, satisfying the following conditions for homogeneous $u, v \in U$:

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large},$$

$$(Du)_n v = -n u_{n-1} v,$$

$$D(u_n v) = (Du)_n v + u_n (Dv),$$

$$|u_n v| = |u| + |v|.$$

Then we shall say that U is a VLA-J-SS (something like a vertex Lie superalgebra without the half Jacobi identity and without the half skew symmetry).

If for an VLA-J-SS U the half Jacobi identity (3.6) holds, then we shall say that U is a VLA-SS.

If for an VLA-SS U the half skew symmetry (3.5) holds, then clearly U is a vertex Lie superalgebra.

For U and W having any of the above structures we define a homomorphism φ to be a $\mathbb{Z}_2$-grading preserving linear map $\varphi : U \to W$ such that

$$\varphi(u_n v) = (\varphi(u))_n(\varphi(v)), \quad \varphi D = D\varphi.$$

Left (resp. right, two-sided) ideals in U are defined as left (resp. right, two-sided) ideals for all multiplications.

Lemma 6.1. *In the definition of VLA-SS the half Jacobi identity can be equivalently substituted by the half commutator formula.*

Proof. Let us assume that the half commutator formula holds:

$$[Y(u,z_1), Y(v,z_2)] \simeq \mathrm{Res}_{z_0}\, z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v, z_2). \tag{6.1}$$

Then for $m \geq 0$ we have

$$\mathrm{Res}_{z_0}\, z_0^m z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v, z_2) \tag{6.2}$$

$$= \mathrm{Res}_{z_0} (z_1 - z_2)^m z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v,z_2) \tag{6.3}$$

$$= (z_1 - z_2)^m \mathrm{Res}_{z_0} z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v,z_2) \tag{6.4}$$

$$\simeq (z_1 - z_2)^m (Y(u,z_1)Y(v,z_2) - \varepsilon_{u,v} Y(v,z_2)Y(u,z_1)) \tag{6.5}$$

$$= (z_1 - z_2)^m \mathrm{Res}_{z_0} z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u,z_1)Y(v,z_2)$$

$$-\varepsilon_{u,v}(z_1 - z_2)^m \mathrm{Res}_{z_0} z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v,z_2)Y(u,z_1) \tag{6.6}$$

$$= \mathrm{Res}_{z_0} z_0^m z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u,z_1)Y(v,z_2)$$

$$-\varepsilon_{u,v} \mathrm{Res}_{z_0} z_0^m z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v,z_2)Y(u,z_1), \tag{6.7}$$

which is precisely the half Jacobi identity.

Here we use the usual properties of δ-function. Since $m \geq 0$, we have a polynomial $z_0^m = ((z_0 - z_1) + z_1)^m$ which may be expanded in powers $(z_0 - z_1)^k z_1^{m-k}$. In the presence of δ-function we may replace a power $(z_0 - z_1)^k$ by $(-z_2)^k$, and as a result we get a polynomial $(z_1 - z_2)^m$. In this way we get (6.3) from (6.2) and (6.7) from (6.6). Since the polynomial $(z_1 - z_2)^m$ is a linear combination of powers $z_1^k z_2^{m-k}$, $k \geq 0$, $m - k \geq 0$, to establish the equality of principal parts of (6.4) and (6.5) it is enough to know the equality of principal parts of formal Laurent series appearing in the half commutator formula (6.1). □

Lemma 6.2. *In the definition of VLA-SS the half Jacobi identity can be equivalently substituted by the half associator formula.*

Proof. Let us assume that the half associator formula holds:

$$Y(Y(u,z_0)v,z_2) \simeq \mathrm{Res}_{z_1} z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u,z_1)Y(v,z_2)$$

$$-\varepsilon_{u,v} \mathrm{Res}_{z_1} z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v,z_2)Y(u,z_1). \tag{6.8}$$

Let $n \geq 0$. By using arguments as in the proof of Lemma 6.1 we get

$$\mathrm{Res}_{z_1} z_1^n z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v,z_2)$$

$$= \mathrm{Res}_{z_1} (z_2 + z_0)^n z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v,z_2)$$

$$= (z_2+z_0)^n Y(Y(u,z_0)v,z_2)$$

$$\simeq (z_2+z_0)^n \operatorname{Res}_{z_1} z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y(u,z_1)Y(v,z_2)$$

$$-\varepsilon_{u,v}(z_2+z_0)^n \operatorname{Res}_{z_1} z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)$$

$$= \operatorname{Res}_{z_1} (z_2+z_0)^n z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)$$

$$-\varepsilon_{u,v} \operatorname{Res}_{z_1} (z_2+z_0)^n z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)$$

$$= \operatorname{Res}_{z_1} z_1^n z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)$$

$$-\varepsilon_{u,v} \operatorname{Res}_{z_1} z_1^n z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1),$$

which is precisely the half Jacobi identity. □

Remark 6.3. The proofs of Lemmas 6.1 and 6.2 are modifications of the proofs of the corresponding statements for vertex superalgebras given in [12, Propositions 2.2.4 and 2.2.6].

Lemma 6.4. *Let U be a VLA-J-SS. Let U_1 be a subspace spanned by vectors of the form*

$$u_m v_n w - \varepsilon_{u,v} v_n u_m w - \sum_{i\ge 0}\binom{m}{i}(u_i v)_{n+m-i} w \tag{6.9}$$

for all homogeneous $u,v,w\in U$ and $m,n\in\mathbb{N}$. Let U_2 be a subspace of U spanned by vectors of the form

$$\sum_{i\ge 0}(-1)^i\binom{k}{i}(u_{m+k-i}(v_{n+i}w) - \varepsilon_{u,v}(-1)^k v_{n+k-i}(u_{m+i}w))$$

$$-\sum_{i\ge 0}\binom{m}{i}(u_{k+i}v)_{m+n-i}w \tag{6.10}$$

for all homogeneous $u,v,w\in U$ and $k,m,n\in\mathbb{N}$. Then $U_1=U_2$.

Proof. For two Laurent series A and B let us write $A\equiv B$ if the coefficients of principal parts of A and B are equal modulo elements in U_1. By definition we have

$$[Y(u,z_1),Y(v,z_2)]w \equiv \operatorname{Res}_{z_0} z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2)w.$$

Then, by almost copying the proof of Lemma 6.1, we get

$$\operatorname{Res}_{z_0} z_0^m z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2)w$$

$$\equiv \operatorname{Res}_{z_0} z_0^m z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)w$$

$$-\varepsilon_{u,v}\operatorname{Res}_{z_0} z_0^m z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)w,$$

which implies $U_2\subset U_1$. It is clear that $U_1\subset U_2$. □

If U is a VLA-J-SS, we shall denote by $\langle\text{Jacobi}\rangle$ the two-sided ideal in U generated by vectors of the form (6.10), or equivalently, by vectors of the form (6.9). It is clear that $\langle\text{Jacobi}\rangle$ is D invariant since D is a derivation for all multiplications. It is also clear that $\langle\text{Jacobi}\rangle$ is a $\mathbb{Z}_2$-graded subspace of U and that we have:

Lemma 6.5. *Let U be a VLA-J-SS and let*

$$U_J = U/\langle\text{Jacobi}\rangle.$$

Then U_J is a VLA-SS with the universal property that any homomorphism φ from U to a vertex Lie superalgebra V factors through U_J by an homomorphism φ_J from U_J to V.

Lemma 6.6. *Let W be a VLA-SS and let $\langle\text{skew symm.}\rangle$ be a subspace spanned by vectors*

$$v_n w+\varepsilon_{v,w}\sum_{k\ge 0}(-1)^{n+k}(D^k/k!)w_{n+k}v \tag{6.11}$$

for all homogeneous $v,w\in W$ and $n\ge 0$. Then $\langle\text{skew symm.}\rangle$ is a $\mathbb{Z}_2$-graded two-sided ideal in W invariant for D. Moreover,

$$W_{ss} = W/\langle\text{skew symm.}\rangle$$

is a VSLA with the universal property that any homomorphism φ from W to a vertex Lie superalgebra V factors through W_{ss} by an homomorphism φ_{ss} from W_{ss} to V.

Proof. Set $W_1=\langle\text{skew symm.}\rangle$. It is clear that $DW_1\subset W_1$ since by assumption D is a derivation for all multiplications.

For two Laurent series A and B let us write $A\equiv B$ if the coefficients of principal parts of A and B are equal modulo elements in W_1. By definition we have

$$e^{-z_2D}Y(v,z_2)w\equiv\varepsilon_{v,w}Y(w,-z_2)v.$$

In order to show that W_1 is a left ideal consider

$$\begin{aligned}
Y(u,z_0)\mathrm{e}^{-z_2D}Y(v,z_2)w &= \mathrm{e}^{-z_2D}Y(u,z_0+z_2)Y(v,z_2)w\\
&= \mathrm{e}^{-z_2D}\,\mathrm{Res}_{z_1}\, z_1^{-1}\delta\left(\frac{z_0+z_2}{z_1}\right)\, Y(u,z_0+z_2)Y(v,z_2)w\\
&= \mathrm{e}^{-z_2D}\,\mathrm{Res}_{z_1}\, z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\, Y(u,z_1)Y(v,z_2)w\\
&\simeq \varepsilon_{u,v}\mathrm{e}^{-z_2D}\,\mathrm{Res}_{z_1}\, z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\, Y(v,z_2)Y(u,z_1)w\\
&\quad+\mathrm{e}^{-z_2D}\,\mathrm{Res}_{z_1}\, z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\, Y(Y(u,z_0)v,z_2)w\\
&= \varepsilon_{u,v}\mathrm{Res}_{z_1}\, z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\,\mathrm{e}^{-z_2D}Y(v,z_2)Y(u,z_1)w+\mathrm{e}^{-z_2D}Y(Y(u,z_0)v,z_2)w\\
&\equiv \varepsilon_{u,v}\varepsilon_{u,v}\varepsilon_{w,v}\,\mathrm{Res}_{z_1}\,(-z_2)^{-1}\delta\left(\frac{z_0-z_1}{-z_2}\right)Y(Y(u,z_1)w,-z_2)v\\
&\quad+\mathrm{e}^{-z_2D}Y(Y(u,z_0)v,z_2)w\\
&\simeq \varepsilon_{v,w}Y(u,z_0)Y(w,-z_2)v-\varepsilon_{u,w}\varepsilon_{v,w}Y(w,-z_2)Y(u,z_0)v\\
&\quad+\mathrm{e}^{-z_2D}Y(Y(u,z_0)v,z_2)w\\
&\equiv \varepsilon_{v,w}Y(u,z_0)Y(w,-z_2)v.
\end{aligned}$$

Hence we have

$$Y(u,z_0)(\mathrm{e}^{-z_2D}Y(v,z_2)w-\varepsilon_{v,w}Y(w,-z_2)v)\equiv 0.$$

This shows that W_1 is a left ideal, i.e., that $u_nW_1\subset W_1$ for all $u\in W$ and $n\geq 0$. It remains to show that W_1 is a right ideal, so consider

$$\begin{aligned}
Y(Y(u,z_0)v,z_2)w &\equiv \varepsilon_{u,w}\varepsilon_{v,w}\,\mathrm{e}^{z_2D}Y(w,-z_2)Y(u,z_0)v &(6.12)\\
&\equiv \varepsilon_{u,w}\varepsilon_{v,w}\,\mathrm{e}^{z_2D}Y(w,-z_2)\varepsilon_{v,u}\mathrm{e}^{z_0D}Y(v,-z_0)u &(6.13)\\
&\equiv \varepsilon_{v,u}Y(\mathrm{e}^{z_0D}Y(v,-z_0)u,z_2)w.
\end{aligned}$$

Hence we have

$$Y(Y(u,z_0)v-\varepsilon_{v,u}\,\mathrm{e}^{z_0D}Y(v,-z_0)u,z_2)w\equiv 0.$$

This shows that W_1 is a right ideal, i.e., that $(W_1)_nw\subset W_1$ for all $w\in W$ and $n\geq 0$. Note that we obtained (6.13) from (6.12) by using the fact that W_1 is a left ideal invariant for D. The remaining statements are clear. $\square$

If U is a VLA-J-SS, we set

$$U_{J+\mathrm{ss}} = (U_J)_{\mathrm{ss}} = U/\langle \mathrm{Jacobi}\rangle/\langle \text{skew symm.}\rangle.$$

Moreover, for any homomorphism φ from U to a vertex Lie superalgebra V we set

$$\varphi_{J+\mathrm{ss}} = (\varphi_J)_{\mathrm{ss}} = \varphi_{\mathrm{ss}} \circ \varphi_J.$$

Then clearly $\varphi : U \to V$ factors through $U_{J+\mathrm{ss}}$ by the homomorphism $\varphi_{J+\mathrm{ss}} : U_{J+\mathrm{ss}} \to V$. Obviously

$$U_{J+\mathrm{ss}} = U/\langle J + \mathrm{ss}\rangle,$$

where $\langle J+\mathrm{ss}\rangle$ is a two-sided ideal in U generated by elements of the form (6.9) and (6.11). It is clear that $\langle J+\mathrm{ss}\rangle$ is D invariant since D is a derivation for all multiplications. It is also clear that $\langle J+\mathrm{ss}\rangle$ is $\mathbb{Z}_2$-graded.

7. Vertex Lie algebras generated by formulas

Let S be a vector space given bilinear maps $F_n^k : S \times S \to S$ for $n, k \geq 0$. Then we can ask under what conditions on F_n^k there is a vertex algebra V containing S and generated by the set of fields $\{Y(u,z) \mid u \in S\}$ for which the commutator formula is

$$[Y(u,z_1), Y(v,z_2)] = \sum_{n,k\geq 0} \frac{(-1)^n}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}z_1}\right)^n z_2^{-1}\delta(z_1/z_2) \left(\frac{\mathrm{d}}{\mathrm{d}z_2}\right)^k Y(F_n^k(u,v), z_2).$$

If there is such V, we could say that V is generated by a formula defined by the set of maps $\{F_n^k \mid n,k\geq 0\}$. There are many examples when this is the case; besides the examples in [12] mentioned before one may consult [3] and the references therein.

We shall say that a $\mathbb{Z}_2$-graded vector space S is a formula if it is equipped with infinitely many $\mathbb{Z}_2$-grading preserving linear maps

$$F_n : S \otimes S \to \mathbb{C}[D] \otimes S, \quad n \in \mathbb{N}$$

with the space $\mathbb{C}[D]$ of formal polynomials in D considered to be even, such that for $u, v \in S$

$$F_n(u,v) = 0 \quad \text{for } n \text{ sufficiently large.} \tag{7.1}$$

We shall write

$$F_n(u,v) = u_n v = \sum_{k\geq 0} D^k \otimes F_n^k(u,v), \quad u, v, F_n^k(u,v) \in S.$$

It is clear that giving the maps F_n for all $n \geq 0$ is equivalent to giving a linear map

$$Y : S \otimes S \to z^{-1}(\mathbb{C}[D]\otimes S)[[z^{-1}]], \qquad Y(u,z)v = \sum_{n\geq 0} F_n(u,v) z^{-n-1}$$

and it would be proper to say that a formula is a given map $Y_S = Y$, or a pair (S, Y), with the prescribed properties.

We define a linear operator $D(D^k \otimes u) = D^{k+1} \otimes u$ and consider $S \subset \mathbb{C}[D] \otimes S$. Then $D^k \otimes u$ can also be written as $D^k u$, and in general elements in $\mathbb{C}[D] \otimes S$ can be written as linear combinations of elements of the form $P(D)u$, where $u \in U$ and $P(D)$ is a polynomial P of the operator D. We shall use the notation $A|_{D=0} = 0$ with the obvious meaning that $A \in D\mathbb{C}[D] \otimes S$.

Example 1. Let $S = S^0 = \boldsymbol{g} \oplus \mathbb{C}c$ be a sum of 1-dimensional space and a Lie algebra $\boldsymbol{g}$ with an invariant bilinear form $\langle \cdot, \cdot \rangle$. Set

$$x_0 y = [x, y], \qquad x_0 c = c_0 x = c_0 c = 0,$$
$$x_1 y = \langle x, y \rangle c, \qquad x_1 c = c_1 x = c_1 c = 0$$

for all $x, y \in \boldsymbol{g}$. Set $F_n = 0$ for $n \geq 2$. Then S is a formula. We could also write

$$Y(x,z)y = \frac{[x, y]}{z} + \frac{\langle x, y \rangle c}{z^2}, \qquad Y(x,z)c = 0, \qquad Y(c,z) = 0.$$

Example 2. Let $S = S^0 = \mathbb{C}\omega \oplus \mathbb{C}c$ be a 2-dimensional space. Set

$$Y(\omega,z)\omega = \frac{D\omega}{z} + \frac{2\omega}{z^2} + \frac{(1/2)c}{z^4}, \qquad Y(\omega,z)c = 0, \qquad Y(c,z) = 0.$$

Then S is a formula.

Lemma 7.1. *Bilinear operations F_n on $S \subset \mathbb{C}[D] \otimes S$ extend in a unique way to $\mathbb{Z}_2$-grading preserving linear maps*

$$F_n : (\mathbb{C}[D] \otimes S) \otimes (\mathbb{C}[D] \otimes S) \to \mathbb{C}[D] \otimes S, \quad n \in \mathbb{N}$$

such that

$$(DA)_n B = -n A_{n-1} B, \tag{7.2}$$

$$D(A_n B) = (DA)_n B + A_n (DB) \tag{7.3}$$

for all $A, B \in \mathbb{C}[D] \otimes S$ and $n \geq 0$, where we write $F_n(A, B) = A_n B$. Moreover, for $A, B \in \mathbb{C}[D] \otimes S$

$$A_n B = 0 \quad \textit{for } n \textit{ sufficiently large.}$$

Proof. First note that our conditions can be written as

$$Y(DA,z)B = (\mathrm{d}/\mathrm{d}z)Y(A,z)B, \qquad Y(A,z)DB = (D - \mathrm{d}/\mathrm{d}z)Y(A,z)B.$$

Hence it must be

$$Y(P(D)u,z)Q(D)v = P(\mathrm{d}/\mathrm{d}z)Q(D - \mathrm{d}/\mathrm{d}z)Y(u,z)v \tag{7.4}$$

for $u, v \in S$ and any polynomials P and Q. It is easy to check all the properties stated. $\square$

In a way Lemma 7.1 means that a formula extends to a VLA-J-SS. If $\varphi: S \to V$ is a linear map from a formula S to a vertex Lie superalgebra V, we extend it to a map

$$\tilde{\varphi}: \mathbb{C}[D] \otimes S \to V, \qquad \tilde{\varphi}(D^k \otimes u) = D^k \varphi(u).$$

We shall say that $\varphi: S \to V$ is a homomorphism from S to V if $\tilde{\varphi}$ is a homomorphism (cf. Section 6). By using (7.4) it is easy to see that a linear map $\varphi: S \to V$ is a homomorphism if and only if

$$\tilde{\varphi}(u_n v) = \varphi(u)_n \varphi(v) \quad \text{for all } u, v \in S.$$

Obviously Lemmas 6.5 and 6.6 imply:

Lemma 7.2. *Any homomorphism $\varphi: S \to V$ from a formula S to a vertex Lie superalgebra V factors through $(\mathbb{C}[D] \otimes S)_{J+\mathrm{ss}}$ by a homomorphism $\tilde{\varphi}_{J+\mathrm{ss}}: (\mathbb{C}[D] \otimes S)_{J+\mathrm{ss}} \to V$.*

We shall say that S is an $(J + \mathrm{ss})$-injective formula if the restriction of the quotient map to S:

$$S \hookrightarrow \mathbb{C}[D] \otimes S \to (\mathbb{C}[D] \otimes S)_{J+\mathrm{ss}}$$

is an injection.

Lemma 7.3. *A formula S is $(J + \mathrm{ss})$-injective if and only if there is a vertex Lie superalgebra V and an injective homomorphism φ from S to V.*

Proof. If S is $(J+\mathrm{ss})$-injective, then take $V = (\mathbb{C}[D] \otimes S)_{J+\mathrm{ss}}$. If $\varphi: S \to V$ is injective, then the restriction on S of the quotient map must be injective. $\square$

In the case when S is $(J+\mathrm{ss})$-injective we can consider S as a subspace of the vertex Lie superalgebra $U = (\mathbb{C}[D] \otimes S)_{J+\mathrm{ss}}$ for which the multiplications $u_n v$ are determined by the formula

$$u_n v = \sum_{k \geq 0} D^k F_n^k(u, v), \quad u, v, F_n^k(u, v) \in S. \tag{7.5}$$

Moreover, for a VLSA V a linear map $\varphi: U_{J+\mathrm{ss}} \to V$ is a homomorphism if and only if

$$\varphi(u)_n \varphi(v) = \sum_{k \geq 0} D^k \varphi(F_n^k(u, v)) \quad \text{for } u, v \in S.$$

In another words, relations (7.5) consistently determine the multiplications $u_n v$ in a vertex Lie superalgebra $U = \operatorname{span}\{D^k w \mid w \in S,\ k \geq 0\}$ if and only if S is $(J + \mathrm{ss})$-injective. What remains to be seen is under what conditions this is the case. We have only a partial answer to this problem which, at least, covers the examples listed before.

The following two lemmas are the consequence of (7.4), some calculations get to be simpler by using [12, Eqs. (2.1.8) and (2.1.15)]:

Lemma 7.4. *For $u,v,w \in S$ and polynomials P, Q and R we have in $\mathbb{C}[D] \otimes S$*

$$\begin{aligned}
&[Y(P(D)u,z_1), Y(Q(D)v,z_2)]R(D)w \\
&\quad -\mathrm{Res}_{z_0} z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y(Y(P(D)u,z_0)Q(D)v,z_2)R(D)w \\
&\quad = P(\mathrm{d}/\mathrm{d}z_1)Q(\mathrm{d}/\mathrm{d}z_2)R(D-\mathrm{d}/\mathrm{d}z_1-\mathrm{d}/\mathrm{d}z_2) \\
&\quad \times\left([Y(u,z_1), Y(v,z_2)]w - \mathrm{Res}_{z_0} z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y(Y(u,z_0)v,z_2)w\right).
\end{aligned}$$

Lemma 7.5. *For $u,v \in S$ and polynomials P and Q we have in $\mathbb{C}[D] \otimes S$*

$$\begin{aligned}
&Y(P(D)u,z)Q(D)v - \varepsilon_{u,v}e^{zD}Y(Q(D)v,-z)P(D)u \\
&\quad = P(\mathrm{d}/\mathrm{d}z)Q(D-\mathrm{d}/\mathrm{d}z)(Y(u,z)v - \varepsilon_{u,v}e^{zD}Y(v,-z)u) \\
&\quad \equiv P(\mathrm{d}/\mathrm{d}z)Q(-\mathrm{d}/\mathrm{d}z)(Y(u,z)v - \varepsilon_{u,v}Y(v,-z)u)|_{D=0},
\end{aligned}$$

where $\equiv$ means that the coefficients in the principal part of Laurent series in z are equal modulo subspace $D\mathbb{C}[D] \otimes S$.

Proposition 7.6. *Let S be a formula and assume that $F_n(u,v) \in S$ for all $u,v \in S$ and $n \geq 0$. Then $\mathbb{C}[D] \otimes S$ is a vertex Lie superalgebra if and only if $F_n = 0$ for all $n \geq 1$ and F_0 defines a Lie superalgebra structure on S.*

Proof. Let $\mathbb{C}[D] \otimes S$ be a VLSA. Then the assumption $F_n(u,v) \in S$ together with the half skew symmetry relation (3.8) implies $F_n = 0$ for all $n \geq 1$ and $u_0v = -\varepsilon_{u,v}v_0u$. The Jacobi identity (3.2) for $k = m = n = 0$ is the Jacobi identity for Lie superalgebras. In the same way we see that the assumption $F_n = 0$ for all $n \geq 1$ and F_0 a Lie superalgebra commutator implies the half Jacobi identity and the half skew symmetry relation for elements in S, so then by Lemmas 7.4 and 7.5 they hold for all elements in $\mathbb{C}[D] \otimes S$. □

Clearly Lemmas 6.1 and 7.4 imply the following:

Proposition 7.7. *Let S be a formula. Then $\mathbb{C}[D] \otimes S$ is a VLA-SS if and only if the half commutator relation* (6.1) *holds in $\mathbb{C}[D] \otimes S$ for all $u,v,w \in S$.*

Note that by Lemmas 6.4 and 7.4 the ideal ⟨Jacobi⟩ in $\mathbb{C}[D] \otimes S$ is a two-sided D invariant ideal generated by elements

$$u_m v_n w - \varepsilon_{u,v} v_n u_m w - \sum_{i \geq 0} \binom{m}{i} (u_i v)_{n+m-i} w \tag{7.6}$$

for all homogeneous $u, v, w \in S$ and $m, n \in \mathbb{N}$. Also note that by Lemmas 6.6 and 7.5 the ideal $\langle$skew symm.$\rangle$ in $(\mathbb{C}[D] \otimes S)_J$ is a two-sided D invariant ideal spanned by elements

$$u_n v + \varepsilon_{u,v} \sum_{k \geq 0} (-1)^{n+k} (D^k/k!) v_{n+k} u \tag{7.7}$$

for all homogeneous $u, v \in S$ and $n \in \mathbb{N}$. Hence it is clear from our construction of

$$(\mathbb{C}[D] \otimes S)_{J+\mathrm{ss}} = (\mathbb{C}[D] \otimes S)/\langle \text{Jacobi} \rangle / \langle \text{skew symm.} \rangle = (\mathbb{C}[D] \otimes S)/\langle J + \mathrm{ss} \rangle$$

that we have

Proposition 7.8. *Let S be a formula such that $\langle J + \mathrm{ss} \rangle \subset D\mathbb{C}[D] \otimes S$, or equivalently, such that*

$$\langle \text{Jacobi} \rangle \subset D\mathbb{C}[D] \otimes S \tag{7.8}$$

and that for all homogeneous $u, v \in S$ and $n \geq 0$

$$u_n v + \varepsilon_{u,v} (-1)^n v_n u \in D\mathbb{C}[D] \otimes S. \tag{7.9}$$

Then S is $(J + \mathrm{ss})$-injective.

Note that condition (7.9) is equivalent to $F_n^0(u, v) = -\varepsilon_{u,v}(-1)^n F_n^0(v, u)$. In Example 1 it reads $[x, y] = -[y, x]$, $\langle x, y \rangle = \langle y, x \rangle$, whereas in Example 2 it is obviously satisfied.

In Example 1 condition (7.8) holds since $\langle \text{Jacobi} \rangle = 0$. This amounts to a verification of relations

$$u_0 v_0 w - v_0 u_0 w = (u_0 v)_0 w,$$

$$u_0 v_1 w - v_1 u_0 w = (u_0 v)_1 w,$$

$$u_1 v_0 w - v_0 u_1 w = (u_0 v)_1 w + (u_1 v)_0 w$$

for $u, v, w \in S$, the relation $u_1 v_1 w - v_1 u_1 w = (u_1 v)_1 w$ and all the other obviously hold.

In Example 2 condition (7.8) holds because $\langle \text{Jacobi} \rangle = D\mathbb{C}[D]c$. In principle we could see this by a direct computation of elements in (7.6), i.e., by a direct computation of the principal part of

$$[Y(\omega, z_1), Y(\omega, z_2)]\omega - \sum_{i=0}^{1} \frac{(-1)^i}{i!} \left(\frac{\mathrm{d}}{\mathrm{d}z_1} \right)^i z_2^{-1} \delta(z_1/z_2) Y(\omega_i \omega, z_2)\omega,$$

but since we know there is a Virasoro vertex operator algebra V of level $\ell \neq 0$, we can argue the other way around: First note that $D\mathbb{C}[D]c$ is a two-sided D invariant ideal in $\mathbb{C}[D] \otimes S$. By a direct computation in $\mathbb{C}[D] \otimes S$ we see that

$$\omega_0 \omega_3 \omega - \omega_3 \omega_0 \omega - (\omega_0 \omega)_3 \omega = -(1/2)Dc,$$

so $\langle\text{Jacobi}\rangle \supset D\mathbb{C}[D]c$. A linear map $\mathbb{C}[D]\otimes S \to V$ defined by $c \mapsto \ell\mathbf{1}$, $\omega \mapsto \omega$ factors through the quotient

$$\mathbb{C}c \oplus \mathbb{C}[D]\omega \cong (\mathbb{C}[D]\otimes S)/D\mathbb{C}[D]c \to \operatorname{span}\{\mathbf{1}, D^k\omega \mid k \geq 0\} \subset V$$

and it is obviously a homomorphism of VLA-J-SS. Since $\{\mathbf{1}, D^k\omega \mid k \geq 0\}$ is linearly independent set in V, this map is an isomorphism, and hence $\langle\text{Jacobi}\rangle = D\mathbb{C}[D]c = D\mathbb{C}[D]\otimes c$.

Example 3. Let $S = S^0 = B \oplus \mathbb{C}c$ be a sum of 1-dimensional space and an algebra $(B, \cdot)$ with a symmetric bilinear form $\langle\cdot,\cdot\rangle$. For $u, v \in B$ set

$$Y(u,z)v = \frac{D(u\cdot v)}{z} + \frac{u\cdot v + v\cdot u}{z^2} + \frac{\frac{1}{2}\langle u,v\rangle c}{z^4}, \qquad Y(u,z)c = 0, \qquad Y(c,z) = 0.$$

It is easy to see that for $u, v, w \in S$ the elements of the form (7.6) and (7.7) are in the ideal $D\mathbb{C}[D]\otimes c$, or equivalently, that $\langle J + \text{ss}\rangle$ is contained in the ideal $D\mathbb{C}[D]\otimes c$, if and only if B is a right Novikov algebra (cf. [15]), i.e. if and only if

$$u\cdot(v\cdot w) = v\cdot(u\cdot w),$$
$$(v\cdot w)\cdot u + v\cdot(u\cdot w) = v\cdot(w\cdot u) + (v\cdot u)\cdot w,$$
$$\langle u\cdot v, w\rangle = \langle v\cdot u, w\rangle = \langle v, u\cdot w\rangle = \langle v, w\cdot u\rangle.$$

If B is an associative commutative algebra with a symmetric associative bilinear form $\langle\cdot,\cdot\rangle$, then all these conditions are clearly satisfied. For an obvious noncommutative example we take a linear functional λ on a vector space B and set $u\cdot v = \lambda(u)v$, $\langle u,v\rangle = \lambda(u)\lambda(v)$. Note that $\lambda(\omega) = 1$ implies $\omega\cdot\omega = \omega$.

If the above conditions are satisfied, then S is $(J+\text{ss})$-injective, that is $S \subset U = (\mathbb{C}[D]\otimes S)_{J+\text{ss}}$. If we set $Y(u,z) = \sum_{n\in\mathbb{Z}} u(n)z^{-n-2}$, then in the corresponding Lie algebra $\mathcal{L}(U)$ we have the commutation relations

$$[u(n), v(m)] = (n+1)(v\cdot u)(n+m) - (m+1)(u\cdot v)(n+m) + \frac{1}{2}\binom{n+1}{3}\delta_{n+m,0}\langle u,v\rangle c.$$

We can summarize our constructions in the following:

Theorem 7.9. *Let (S, Y_S) be a formula given by*

$$Y_S(u,z)v = \sum_{n,k\geq 0} \frac{D^k \otimes F_n^k(u,v)}{z^{n+1}}$$

and assume that $\langle\text{Jacobi}\rangle|_{D=0} = 0$ *and* $F_n^0(u,v) = -\varepsilon_{u,v}(-1)^n F_n^0(v,u)$ *for* $u, v \in S$ *and* $n \geq 0$.

Then there exists a vertex superalgebra V and an injective homomorphism $S \hookrightarrow V$ such that V is generated by the set of fields $\{Y(u,z) \mid u \in S\}$ for which the commutator formula is

$$\begin{aligned}&[Y(u,z_1), Y(v,z_2)]\\ &\quad = \sum_{n,k\geq 0} \frac{(-1)^n}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}z_1}\right)^n z_2^{-1}\delta(z_1/z_2)\left(\frac{\mathrm{d}}{\mathrm{d}z_2}\right)^k Y(F_n^k(u,v),z_2).\end{aligned} \tag{7.10}$$

Among all such vertex superalgebras, i.e., $S \hookrightarrow V$ to be precise, there exists a vertex superalgebra $\mathscr{V}(\langle S\rangle)$ such that any other V is a quotient of $\mathscr{V}(\langle S\rangle)$. Moreover, for any V as above we have

$$Y_V(u,z)v \simeq \sum_{n,k\geq 0} \frac{D^k F_n^k(u,v)}{z^{n+1}}. \tag{7.11}$$

Proof. By Proposition 7.8 the formula S is $(J + \mathrm{ss})$-injective. So consider $S \subset \langle S\rangle$, where $\langle S\rangle = (\mathbb{C}[D] \otimes S)_{J+\mathrm{ss}}$ is a vertex Lie algebra. Note that by our construction on the quotient $\langle S\rangle$ we have a relation

$$u_n v = \sum_{k\geq 0} D^k F_n^k(u,v) \tag{7.12}$$

for $u, v \in S$, with $F_n^k(u,v) \in S$. Let $\mathscr{V}(\langle S\rangle)$ be the universal enveloping vertex algebra of $\langle S\rangle$ and, as usual, consider $S \subset \langle S\rangle \subset \mathscr{V}(\langle S\rangle)$. Since $\langle S\rangle$ is generated by S and D, vertex superalgebra $\mathscr{V}(\langle S\rangle)$ is generated by $\{Y(u,z) \mid u \in S\}$ and (7.10) is just the commutator formula in which the product $u_n v$ in $\langle S\rangle \subset \mathscr{V}(\langle S\rangle)$ is expressed by using (7.12). Since on $\langle S\rangle \subset \mathscr{V}(\langle S\rangle)$ the operations for $n \geq 0$ coincide, (7.11) follows as well.

Now assume that V is another vertex superalgebra and that $\varphi : S \hookrightarrow V$ is an injective homomorphism from S to V. By Lemma 7.2 the homomorphism φ factors through $\langle S\rangle$ by an homomorphism $\tilde{\varphi}_{J+\mathrm{ss}} : \langle S\rangle \to V$. By the universal property of $\mathscr{V}(\langle S\rangle)$ this homomorphism $\tilde{\varphi}_{J+\mathrm{ss}}$ extends to a homomorphism $\mathscr{V}(\langle S\rangle) \to V$, and it is surjective since S generates V. □

We may illustrate Theorem 7.9 on the Virasoro formula S given in Example 2. We saw that $U = (\mathbb{C}[D] \otimes S)_{J+\mathrm{ss}}$ is generated by S and that $\langle S\rangle = U = \mathbb{C}c \oplus \mathbb{C}[D]\omega$ with $Dc = 0$. The corresponding Lie algebra $\mathscr{L}(U)$ is the Virasoro Lie algebra

$$\mathscr{L}(U) = \mathbb{C}c \oplus \operatorname{span}\{L(n) \mid n \in \mathbb{Z}\}$$

with a central element c and the commutation relations

$$[L(n), L(m)] = (n-m)L(n+m) + \frac{1}{2}\binom{n+1}{3}\delta_{n+m,0}c.$$

Here we use the usual notation

$$Y(\omega,z) = \sum_{n\in\mathbb{Z}} \omega_n z^{-n-1} = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2},$$

i.e., $L(n)=\omega_{n+1}$, and $c=c_{-1}$, $c_n=0$ for $n\neq -1$. We also have

$$\mathscr{L}_-(U)=\mathbb{C}c\oplus \operatorname{span}\{L(n)\,|\,n\leq -2\},\qquad \mathscr{L}_+(U)=\operatorname{span}\{L(n)\,|\,n\geq -1\}.$$

The derivation D of the Lie algebra $\mathscr{L}(U)$, defined by $D(u_n)=(Du)_n$, equals $\operatorname{ad}(L(-1))$, i.e.,

$$D(L(n))=-(n+1)L(n-1)=[L(-1),L(n)],\quad D(c)=0.$$

By construction, the universal enveloping vertex algebra $\mathscr{V}(\langle S\rangle)$ of the vertex Lie algebra $U=\langle S\rangle$ can be viewed as the universal enveloping algebra $\mathscr{U}(\mathscr{L}_-(U))$ of the Lie algebra $\mathscr{L}_-(U)$. Under this identification $\mathbf{1}$ is the identity and D is a derivation of the associative algebra $\mathscr{U}(\mathscr{L}_-(U))$ extending the derivation D of the Lie algebra $\mathscr{L}_-(U)$. The vertex algebra $\mathscr{V}(\langle S\rangle)$ is generated by fields

$$Y(\omega,z),\qquad Y(c,z)=c\quad\text{and}\quad Y(\mathbf{1},z)=\mathrm{id}_{\mathscr{V}(\langle S\rangle)}.$$

Note that c acts as a left multiplication by c, and that $D=L(-1)$.

For any vertex operator superalgebra V of level ℓ with a conformal vector ω we have an injective homomorphism $S\hookrightarrow V$, defined by $c\mapsto \ell\mathbf{1}$, $\omega\mapsto\omega$, which extends to a homomorphism $\mathscr{V}(\langle S\rangle)\to V$ of vertex superalgebras. Clearly $\langle c-\ell\mathbf{1}\rangle=(c-\ell\,\mathrm{id})\mathscr{V}(\langle S\rangle)$ is an ideal in $\mathscr{V}(\langle S\rangle)$, so we have a vertex superalgebra homomorphism

$$\mathscr{V}(\langle S\rangle)/\langle c-\ell\mathbf{1}\rangle\to V.$$

8. Conformal vectors

In the previous section we have considered three examples, and in each of them the corresponding vertex Lie algebra $U=(\mathbb{C}[D]\otimes S)_{J+\mathrm{ss}}$ is a quotient of $\mathbb{C}[D]\otimes S$ by the ideal $\langle J+\mathrm{ss}\rangle=D\mathbb{C}[D]\otimes c$. In this section we will show that this holds in general for certain class of formulas, roughly speaking the ones corresponding to vertex operator superalgebras. For this reason we need the notions of graded formulas and conformal vectors.

Let (S,Y) be a formula and $d:S\to S$ an even linear map such that for $i\in\mathbb{Z}_2$

$$S^i=\bigoplus_{\lambda\in\mathbb{R}_+}S^i_\lambda,\quad dv=\lambda v\quad\text{for } v\in S^i_\lambda.$$

We extend d to an even linear map on $\mathbb{C}[D]\otimes S$ by setting

$$d(D^kv)=(\lambda+k)v\quad\text{for } v\in S_\lambda,\ k\geq 0.$$

Here we write $S_\lambda=S^0_\lambda+S^1_\lambda$. So the space $\mathbb{C}[D]\otimes S$ is graded by eigenspaces of d:

$$\mathbb{C}[D]\otimes S=\bigoplus_{\mu\in\mathbb{R}_+}(\mathbb{C}[D]\otimes S)_\mu,$$

$$(\mathbb{C}[D]\otimes S)_\mu=\bigoplus_{k\geq 0,\lambda\geq 0,k+\lambda=\mu}D^k\otimes S_\lambda. \tag{8.1}$$

We shall say that the formula (S,Y) is graded by d if

$$[d,Y(u,z)]v=\left(z\frac{\mathrm{d}}{\mathrm{d}z}+\lambda\right)Y(u,z)v \quad \text{for } u\in S_\lambda,\ v\in S. \tag{8.2}$$

Lemma 8.1. *For $A\in(\mathbb{C}[D]\otimes S)_\lambda$ and $B\in(\mathbb{C}[D]\otimes S)_\mu$ we have*

$$A_nB\in(\mathbb{C}[D]\otimes S)_{\lambda+\mu-n-1}. \tag{8.3}$$

Proof. By assumption (8.2) the statement (8.3) holds for $A=u$ and $B=v$ in S. Since

$$(DA)_nB=-nA_{n-1}B$$

and

$$D(\mathbb{C}[D]\otimes S)_\lambda\subset(\mathbb{C}[D]\otimes S)_{\lambda+1}, \tag{8.4}$$

we see by induction that (8.3) holds for $A\in\mathbb{C}[D]\otimes S$ and $B=v\in S$. Now the lemma follows by induction for all B by using (8.4) and (7.3), i.e.,

$$A_n(DB)=(DA)_nB-D(A_nB). \qquad \square$$

Note that (8.4) is equivalent to

$$[d,D]=D. \tag{8.5}$$

Let (S,Y) be a formula and let ω and c be two even nonzero elements in S. We shall say that ω is a conformal vector in S with a central element c if

$$Y(\omega,z)\omega=\frac{D\omega}{z}+\frac{2\omega}{z^2}+\frac{(1/2)c}{z^4},$$

$$Y(c,z)v=0, \quad Y(v,z)c=0 \quad \text{for all } v\in S, \tag{8.6}$$

and if the formula S is graded by a map d such that

$$S_0=\mathbb{C}c, \qquad \dim S_\lambda<\infty \quad \text{for } \lambda\in\mathbb{R}_+,$$

$$Y(\omega,z)v=\frac{Dv}{z}+\frac{dv}{z^2}+\frac{0}{z^3}+\cdots \quad \text{for } v\in S. \tag{8.7}$$

Clearly (8.6) implies the following:

Lemma 8.2. *$D\mathbb{C}[D]\otimes c$ is a two-sided D invariant ideal in $\mathbb{C}[D]\otimes S$.*

Lemma 8.3. *On $\mathbb{C}[D]\otimes S$ we have $\omega_0=D$ and $\omega_1=d$.*

Proof. Let $v \in S$. By assumption (8.7) we have $\omega_0 v = Dv$ and $\omega_1 v = dv$, and for $u = D^k v$, $k \geq 1$, relation (7.4) implies

$$Y(\omega,z)D^k v = \left(D - \frac{\mathrm{d}}{\mathrm{d}z}\right)^k \left(\frac{Dv}{z} + \frac{dv}{z^2} + \cdots\right)$$

$$= \frac{D^{k+1}v}{z} + \frac{D^k dv + kD^k v}{z^2} + \cdots.$$

Hence $\omega_0 D^k v = DD^k v$, and (8.5) implies $\omega_1 D^k v = dD^k v$. $\square$

Theorem 8.4. *Let (S, Y) be a formula and let ω be a conformal vector in S with a central element c. Then the following three conditions are equivalent*:

(i) *S is $(J + \mathrm{ss})$-injective.*

(ii) $\langle J + \mathrm{ss}\rangle = D\mathbb{C}[D] \otimes c$.

(iii) *The elements of the form*

$$u_m v_n w - \varepsilon_{u,v} v_n u_m w - \sum_{i \geq 0} \binom{m}{i} (u_i v)_{n+m-i} w, \tag{8.8}$$

$$v_n w + \varepsilon_{v,w} \sum_{k \geq 0} (-1)^{n+k} (D^k/k!) w_{n+k} v, \tag{8.9}$$

are in $D\mathbb{C}[D] \otimes c$ for all homogeneous $u, v, w \in S$ and $m, n \in \mathbb{N}$.

Proof. (i)$\Rightarrow$(ii): Let S be $(J + \mathrm{ss})$-injective and let $U = (\mathbb{C}[D] \otimes S)_{J+\mathrm{ss}} = \sum_{k \geq 0} D^k S$. As usual set $Y_U(\omega, z) = \sum_{n \geq 0} \omega_n z^{-n-1}$. Then for $v \in S_\lambda \subset U$ and $k \geq 1$ our assumptions imply $\omega_0 = D$ and $\omega_1(D^k v) = (\lambda + k)D^k v$.

The commutator formula for $\omega_0, \omega_1, \omega_2$ shows that on U we have a representation of the Lie algebra sl_2, in standard notation for basis elements

$$e = -\omega_2, \qquad h = -2\omega_1, \qquad f = \omega_0 = D.$$

By our assumptions $v \in S_\lambda \backslash \{0\}$ is a highest weight vector of h-weight -2λ, so $\mathbb{C}[D]v_\lambda$ is an irreducible Verma module for $\lambda > 0$. Hence

$$U = \mathbb{C}[D]c + \left(\bigoplus_{k \geq 0, \lambda > 0} D^k S_\lambda\right),$$

$$\dim(D^k S_\lambda) = \dim S_\lambda \quad \text{for } \lambda > 0. \tag{8.10}$$

Since in $\mathbb{C}[D] \otimes S$ we have $\omega_0 \omega_3 \omega - \omega_3 \omega_0 \omega - (\omega_0 \omega)_3 \omega = -(1/2)Dc \in \langle \text{Jacobi} \rangle$, on the quotient U we have $Dc = 0$ and hence

$$U = \mathbb{C}c \oplus \left(\bigoplus_{k \geq 0, \lambda > 0} D^k S_\lambda\right).$$

In particular, $U = \bigoplus_{\mu \geq 0} U_\mu$ is graded by eigenspaces of ω_1:

$$U_0 = \mathbb{C}c, \quad U_\mu = \bigoplus_{k \geq 0, \lambda > 0, k+\lambda = \mu} D^k S_\lambda \quad \text{for } \mu > 0. \tag{8.11}$$

Since the two-sided ideal $\langle J + \mathrm{ss} \rangle$ in $\mathbb{C}[D] \otimes S$ is generated by elements of the form (8.8) and (8.9), Lemma 8.1 implies that $\langle J + \mathrm{ss} \rangle$ is graded, i.e., invariant for $\omega_1 = d$, and hence

$$U_\mu \cong (\mathbb{C}[D] \otimes S)_\mu / \langle J + \mathrm{ss} \rangle_\mu. \tag{8.12}$$

It follows from (8.1), (8.10) and (8.11) that

$$\dim U_\mu = \dim(\mathbb{C}[D] \otimes S)_\mu - 1 \quad \text{for } \mu > 0, \ \mu \in \mathbb{N},$$

$$\dim U_\mu = \dim(\mathbb{C}[D] \otimes S)_\mu \quad \text{for } \mu > 0, \ \mu \notin \mathbb{N},$$

so (8.12) implies $\dim \langle J + \mathrm{ss} \rangle_\mu = 1$ for $\mu > 0$, $\mu \in \mathbb{N}$, and $\dim \langle J + \mathrm{ss} \rangle_\mu = 0$ for $\mu > 0$, $\mu \notin \mathbb{N}$. Hence $D\mathbb{C}[D] \otimes c \subset \langle J + \mathrm{ss} \rangle$ implies that

$$\langle J + \mathrm{ss} \rangle = D\mathbb{C}[D] \otimes c.$$

(ii) $\Rightarrow$ (iii): This is obvious since by definition $\langle J + \mathrm{ss} \rangle$ is generated by elements of the form (8.8) and (8.9).

(iii) $\Rightarrow$ (i). Since $D\mathbb{C}[D] \otimes c$ is a two-sided ideal in $\mathbb{C}[D] \otimes S$, assumption (iii) clearly implies $\langle J + \mathrm{ss} \rangle \subset D\mathbb{C}[D] \otimes c$. Hence (i) follows by Proposition 7.8. □

Remark 8.5. As a conclusion we could say that in the case of $(J + \mathrm{ss})$-injective formulas S with a conformal vector ω it is enough to consider a VLA-J-SS U of the form

$$U = \mathbb{C}c \oplus (\mathbb{C}[D] \otimes S'), \qquad S' = \bigoplus_{\lambda > 0} S_\lambda,$$

defined by a map Y on $S' \times S'$

$$Y(u,z)v = \sum_{n \geq 0} \frac{u_n v}{z^{n+1}}, \quad u, v \in S', \ u_n v \in U,$$

Y being extended to $U \times U$ by

$$Y(P(D)u,z)Q(D)v = P(\mathrm{d}/\mathrm{d}z)Q(D - \mathrm{d}/\mathrm{d}z)Y(u,z)v,$$

$$Dc = 0 \quad \text{and} \quad D = D \otimes 1 \quad \text{on } \mathbb{C}[D] \otimes S',$$

and check whether it is a vertex Lie algebra, i.e., check whether the elements of the form (8.8) and (8.9) are zero for all $u, v, w \in S'$.

In this case for any given $\ell \in \mathbb{C}$ the quotient $\mathscr{V}_\ell(U) = \mathscr{V}(U)/\langle c - \ell \mathbf{1} \rangle$ of the universal enveloping vertex superalgebra $\mathscr{V}(U)$ is a vertex operator superalgebra with a conformal vector $\omega \in S_2 \subset \mathscr{V}_\ell(U)$.

Example 4. The only possible examples of $(J+\mathrm{ss})$-injective formulas S with a conformal vector ω such that $S=S^0=\mathbb{C}c\oplus S_2$ are a special case of Example 3: $B=S_2$ is a finite dimensional associative commutative algebra with the identity ω and a symmetric associative bilinear form such that $\langle\omega,\omega\rangle=1$. For $u,v\in B$ the formula

$$Y(u,z)v=\frac{D(u\cdot v)}{z}+\frac{2u\cdot v}{z^2}+\frac{\frac{1}{2}\langle u,v\rangle c}{z^4}$$

implies in the corresponding Lie algebra $\mathscr{L}(U)$ the commutation relations

$$[u(n),v(m)]=(n-m)(u\cdot v)(n+m)+\frac{1}{2}\binom{n+1}{3}\delta_{n+m,0}\langle u,v\rangle c.$$

A somewhat different construction of this Lie algebra and the corresponding vertex operator algebra is given in [10]. See also [3] and the references therein.

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After this work was completed I have learned that the notion of vertex Lie algebra is a special case of a more general notion of local vertex Lie algebra introduced and studied in [3] by C. Dong, H.-S. Li and G. Mason. In part their work contains some results similar to the ones obtained in Sections 4 and 5. I thank the authors for sending me their paper and for informing me about some references in physics literature.

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